ON ABA-GROUPS OF FINITE ORDER([1])

BY
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1. **Introduction.** If a group G contains subgroups A and B such that every element $g \in G$ can be written in the form $g=aba'$ with $a, a' \in A$ and $b \in B$, then we say that G is an ABA-group and write $G=ABA$. In this paper we shall determine the structure of finite groups $G=ABA$ in which A and B have relatively prime orders, A is abelian, and one of the following holds:

I. B is nilpotent and A is its own normalizer in G.

II. B is abelian of odd order.

III. B is abelian and there exists an involution in B which normalizes A and whose fixed points on A are contained in the center of G.

Groups satisfying these conditions will be referred to as groups of types I, II, and III, respectively. The most interesting groups of type III are those in which the given involution inverts the elements of A. The simple groups $PSL(2, 2^n)$ are of this form with A of order 2^n+1 and B elementary abelian of order 2^n.

The main results of this paper are the following.

THEOREM. *Every group of type* I *is solvable.*

THEOREM. *Every group of type* II *is solvable.*

THEOREM. *If* $G=ABA$ *is of type* III, *then* $G=A_1B_1A_1 \times A_2B_2A_2 \times \cdots \times A_nB_nA_n$ *where*

(1) $A_i \subseteq A$ *and* $B_i \subseteq B$, $1 \leqq i \leqq n$,

(2) $A_1B_1A_1$ *is either trivial or solvable, and*

(3) *either* $A_iB_iA_i=1$, $2 \leqq i \leqq n$, *or* $A_iB_iA_i \simeq PSL(2, 2^{m_i})$, $m_i \geqq 2$, $2 \leqq i \leqq n$.

It was known that ABA-groups in which A and B are cyclic and either A is its own normalizer in G, or A and B are of relatively prime orders, are solvable [7], [8].

It will be convenient to fix certain notation at the outset. If $G=ABA$, then N and B_0 will denote $N_G(A)$ and $B \cap N_G(A)$ respectively. If $G=ABA$ is of type III, then b_0 will denote an involution in B_0 satisfying $C_A(b_0) \subseteq Z(G)$. If $\bar{H}$ is a homomorphic image of the group H, $x \in H$, and K is a subgroup of H, then $\bar{x}$ and $\bar{K}$ will denote the images in $\bar{H}$ of x and K respectively.

Let A and T be groups and suppose there exists a homomorphism $\phi: A \to \mathrm{Aut}(T)$. Then we shall say that A acts on T or that T is acted on by A.

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For $t \in T$ and $a \in A$, we shall denote $\phi(a)(t)$ by t^a. If X and Y are subsets of A and T respectively, then $C_Y(X)=\{y \in Y \mid y^x=y \text{ for all } x \in X\}$, $C_X(Y)=\{x \in X \mid y^x=y \text{ for all } y \in Y\}$, and $Y^X=\{y^x \mid y \in Y, x \in X\}$. Note in particular that Y^X need not be a group. Note also that although A need not be a group of automorphisms of T, $A/C_A(T)$ may be regarded as such. We shall apply all the terminology of groups of automorphisms to A.

In all other respects, the notation and terminology used in this paper will be the same as that used in [4]. All groups considered here are assumed to be finite.

2. **Assumed results.** We shall several times be able to utilize two recent classification theorems. We list these below together with lemmas covering the information we shall need about the groups $PSL(2, q)$.

THEOREM 2.1 (FEIT-THOMPSON [3]). *All finite groups of odd order are solvable.*

This theorem will be used throughout the paper without specific reference to it.

In obtaining our results on groups of types II and III, we shall need the following additional results.

THEOREM 2.2 (GORENSTEIN [5]). *Let G be a finite group with abelian Sylow 2-subgroups in which the centralizers of involutions are solvable. Then either G is solvable, or $G/O(G)$ is isomorphic to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$ where either $q \equiv 3$ or $5 \pmod 8$, $q \geqq 5$, or $q=2^n$, $n \geqq 2$.*

LEMMA 2.3. *Let $H=PSL(2, q)$ and $H_1=P\Gamma L(2, q)$ where $q=2^n$, $n \geqq 2$. Then:*

(i) *H is simple of order $q(q^2-1)$.*

(ii) *A Sylow 2-subgroup S of H is elementary abelian of order q. S is disjoint from its conjugates. $N_H(S)$ is of order $q(q-1)$ and contains a cyclic group of order $q-1$ which acts transitively on the involutions of S. $C_{H_1}(S)=S$. If T is a nontrivial subgroup of S, then $C_H(T)=S$.*

(iii) *H contains cyclic subgroups R_1 and R_2 of orders $q-1$ and $q+1$ respectively. $C_{H_1}(R_2)=R_2$. $|N_H(R_i)|=2|R_i|$, $i=1, 2$. If $x \in R_i^{\#}$, then $C_H(x)=R_i$, $i=1, 2$. R_i is disjoint from its conjugates, $i=1, 2$.*

(iv) *$H_1=HF$ with $H \triangleleft H_1$, F cyclic of order n, and $H \cap F=1$. F normalizes subgroups of H of orders q, $q-1$, $q+1$. If $F_0 \subseteq F$, then a Sylow 2-subgroup of HF_0 is abelian if and only if $|F_0|$ is odd.*

(v) *If G is a central extension of H, then either $G=H \times Z(G)$, or $H=PSL(2, 4)$, $G=Z(G)L$ where $L \cong SL(2, 5)$, $|Z(G) \cap L|=2$, and Sylow 2-subgroups of G are not abelian.*

(vi) *If H is isomorphic to a normal subgroup M of a group K in which $C_K(M)=1$, then K is isomorphic to a subgroup of H_1 containing H.*

LEMMA 2.4. *Let $H=PSL(2, q)$, $H_1=PGL(2, q)$, and $H_3=P\Gamma L(2, q)$ where $q=p^m$, p an odd prime. Then:*

(i) *H_3 contains a normal subgroup $H_2 = P\Omega L(2, q)$ such that $H_2 \supseteq H_1$ and H_2/H_1 is a 2-complement in H_3/H_1.*

(ii) $|H_1| = 2|H| = q(q^2-1)$. *H is simple if $q > 3$. If $q \equiv 3$ or 5 (mod 8), then $H_2 = H_3$ and a Sylow 2-subgroup of H is of ord?r 4. $PSL(2, 5) \cong PSL(2, 4)$ and $P\Gamma L(2, 5) = PGL(2, 5) \cong P\Gamma L(2, 4)$.*

(iii) *If x is an involution in H, then $C_H(x)$ is dihedral of order $q-d$ where $d \equiv q$ (mod 4), $d = \pm 1$.*

(iv) *H contains cyclic Hall subgroups of order $(q-d)/2^a$ where 2^a is the exact power of 2 dividing $q-d$. Two such subgroups of the same order are conjugate and distinct conjugates have trivial intersections.*

(v) *If x is of odd order dividing $q \pm d$, then $C_H^*(x) = \{h \in H \mid x^h = x \text{ or } x^h = x^{-1}\}$ is of order $q \pm d$. If x is of order dividing $q-d$, then $C_H^*(x) = C_H(y)$ for some involution $y \in H$.*

(vi) *Sylow r-subgroups of H are abelian if r is an odd prime.*

(vii) *Let T be an elementary abelian subgroup of H_1 of order 4. Then $C_{H_2}(T) = T \times F$ where F is a complement to H_1 in H_2.*

For proofs of the statements in Lemmas 2.3 and 2.4, the reader is referred to [1], [2], [10], [9, Lemmas 3.1 and 3.3], and [5, Lemmas 2.1 and 2.2].

3. **Groups of type I.** In this section we shall show that groups of type I are solvable. We first establish some elementary facts about ABA-groups in general.

LEMMA 3.1. *Let $G = ABA$.*

(i) *If H is a subgroup of G and $A \subseteq H$, then $H = A(B \cap H)A$.*

(ii) *$N = AB_0$.*

Proof of (i). Let $h \in H$. Since $G = ABA$, $h = aba'$, $a, a' \in A$, $b \in B$. Then

$$b = a^{-1}ha'^{-1} \in B \cap H.$$

Since this is true for all $h \in H$, $H = A(B \cap H)A$.

Proof of (ii). Let $x \in N$. It follows from (i) that $x = aba' = a(a')^{b^{-1}}b$, $a, a' \in A$, $b \in B \cap N = B_0$. Since $b \in N$, $a(a')^{b^{-1}} \in A$. Therefore $x \in AB_0$. Since this is true for all $x \in X$, $N \subseteq AB_0$. Clearly $AB_0 \subseteq N$. Therefore $N = AB_0$.

We next obtain information about the structure of $\pi(A)$-separable ABA-groups.

THEOREM 3.2. *Let $G = ABA$ with A abelian, B nilpotent, and $(|A|, |B|) = 1$. If G is $\pi(A)$-separable, then $G = (AB_0)T$ where $T = (B \cap T)^A \lhd G$. In particular, $|G|$ divides $|A|\,|B|^n$ for a suitable integer $n \geqq 1$ and A is a Hall subgroup of G.*

Proof. By induction on $|G|$. Let $\pi = \pi(A)$. Let $p \in \pi$, P be a Sylow p-subgroup of A, and $N^* = N_G(P)$. Then $A \subseteq N^*$ and by 3.1(i), $N^* = A(B \cap N^*)A$. Suppose first that $N^* \subset G$. By induction, A is a Hall subgroup of N^* and therefore P is a Sylow p-subgroup of N^*. Then P is a Sylow p-subgroup of G. Suppose next that $N^* = G$. Let $\tilde{G} = G/P$. Then $\tilde{G} = \tilde{A}\tilde{B}\tilde{A}$. By induction, $|\tilde{G}|$ divides $|\tilde{A}|\,|\tilde{B}|^m$ for a suitable integer $m \geqq 1$. Since $(|A|, |B|)=1$ and P is a Sylow p-subgroup of A, p does not divide $|\tilde{G}|$. Therefore P is a Sylow p-subgroup of G. Since this is true for all $p \in \pi$, A is a Hall subgroup of G.

Let $T=O_{\pi'}(G)$ and set $\bar{G}=G/T$. Then $\bar{G}=\bar{A}\bar{B}\bar{A}$. Since G is π-separable, $C_{\bar{G}}(O_\pi(\bar{G})) \subseteq O_\pi(\bar{G})$ [4, Theorem 6.3.2]. Now $\bar{A} \subseteq \bar{A}O_\pi(\bar{G})$. Therefore $\bar{A}O_\pi(\bar{G})=\bar{A}(\bar{B} \cap \bar{A}O_\pi(\bar{G}))\bar{A}$. Since $(|A|, |B|)=1$, $\bar{B} \cap \bar{A}O_\pi(\bar{G})=1$. Therefore $\bar{A}O_\pi(\bar{G}) \subseteq \bar{A}$. Therefore $\bar{A} \subseteq C_{\bar{G}}(O_\pi(\bar{G})) \subseteq O_\pi(\bar{G}) \subseteq \bar{A}$. Therefore $\bar{A} \lhd \bar{G}$. Let $g \in G$. Then $A^g \subseteq AT$. Now A and A^g are abelian Hall subgroups of AT. Therefore $A^g=A^h$ with $h \in AT$ [4, Exercise 6.2]. Therefore $gh^{-1} \in N$ and $g \in N(AT)=NT$. Therefore $G=NT$ and by 3.1(ii), $G=(AB_0)T$.

Let $t \in T$. Since $G=ABA$, $t=aba'$ with $a', a \in A$, $b \in B$. Since $t \in T$, $\bar{t}=\bar{a}\bar{b}\bar{a}'=1$. Therefore $\bar{b}=\bar{a}^{-1}\bar{a}'^{-1} \in \bar{A} \cap \bar{B}=1$. Thus $aa', b \in T$. Since $(|A|, |T|)=1$, $aa'=1$. Thus $t=a'^{-1}ba' \in (B \cap T)^A$. Since this is true for all $t \in T$, $T \subseteq (B \cap T)^A$. On the other hand, since $T \lhd G$, $(B \cap T)^A \subseteq T$. Therefore $T=(B \cap T)^A$.

If $t \in T$, then $|t|$ divides $|B|$. Therefore $|T|$ divides $|B|^{n-1}$ for a suitable integer $n \geqq 1$. Since $G=(AB_0)T$, $|G|$ divides $|A|\,|B|\,|T|$. Therefore $|G|$ divides $|A|\,|B|^n$.

THEOREM 3.3. *Let T be a group acted on by a group A. Suppose $T=D^A$ where D is a nilpotent subgroup of T such that* $(|A|, |D|)=1$. *Then T is nilpotent.*

Proof. Assume false and let $T=D^A$ be a counterexample of minimal order. Let $A_0=C_A(T)$. Then $\bar{A}=A/A_0$ acts on T, $T=D^{\bar{A}}$. and $C_{\bar{A}}(T)=1$. By replacing A by $\bar{A}$, we may assume that $C_A(T)=1$. Since $T=D^A$ and $(|A|, |D|)=1$, $(|A|, |T|)=1$.

We first show that T is solvable. Suppose that T has no nontrivial, A-invariant proper normal subgroups. Since T is not nilpotent, T is not a 2-group. Let p be an odd prime dividing $|T|$. Since $(|A|, |T|)=1$, one of A and T is of odd order and is therefore solvable. Hence A leaves a Sylow p-subgroup S of T invariant [4, Theorem 6.2.2]. Let $J(S)$ be the Thompson subgroup of S and $M=N_T(Z(J(S)))$. Since $J(S)$ char S, $J(S)$ is A-invariant. Hence $M \subset T$, and M is A-invariant. Since M is A-invariant, $M=(D \cap M)^A$. By the minimality of T, M is nilpotent and therefore M has a normal p-complement. Since p is odd, T has a normal p-complement K by the Glauberman-Thompson Theorem [4, Theorem 8.3.1]. Then K char T and therefore K is A-invariant. Hence $K=1$ and $T=P$ is nilpotent, a contradiction. Therefore T possesses an A-invariant proper normal subgroup $H \neq 1$. Since H is A-invariant, $H=(D \cap H)^A$. Furthermore, A acts on $\bar{T}=T/H$ and under this action $\bar{T}=\bar{D}^A$. By the minimality of T, H and $\bar{T}$ are nilpotent. Therefore T is solvable.

Suppose T has two nontrivial, A-invariant normal subgroups N_1 and N_2 such that $N_1 \cap N_2=1$. Then, by the minimality of T, $\bar{T}_i=T/N_i$ is nilpotent, $i=1, 2$. Hence $\bar{T}_1 \times \bar{T}_2$ is nilpotent. The map $\phi(x)=(xN_1, xN_2)$ defines an embedding of T into $\bar{T}_1 \times \bar{T}_2$. Therefore T is nilpotent, a contradiction.

Let P be a minimal A-invariant normal subgroup of T. Since T is solvable, P is an elementary abelian p-group for some prime p. Also $\bar{T}=T/P$ is nilpotent. Since T is not nilpotent, $\bar{T}$ is not a p-group. Let $\bar{Q}' \neq 1$ be a Sylow q-subgroup of $\bar{T}$ invariant under A, $q \neq p$. Let $\bar{Q}$ be a minimal A-invariant subgroup of $\Omega_1(Z(\bar{Q}'))$. Then $\bar{Q} \lhd \bar{T}$ since $\bar{T}$ is nilpotent. Let U be the inverse image of $\bar{Q}$ in T. Then $U = PQ$ where $Q \cong \bar{Q}$ is an elementary abelian q-group, $Q \neq 1$, U is A-invariant, and $U \lhd T$. Since $(|A|, |U|) = 1$, A leaves some Sylow q-subgroup of U invariant. By replacing Q by a conjugate, we may assume that Q is A-invariant. By our choice of $\bar{Q}$, A acts irreducibly on Q.

Suppose $U \subset T$. Then U is nilpotent and therefore $U = P \times Q$. Then Q char U and therefore $Q \lhd T$. Thus we have $P \lhd T$, $Q \lhd T$, $P \cap Q = 1$, and P and Q are A-invariant. This we have seen to be impossible. Therefore $T = U = PQ$. Since T is not nilpotent, it follows that $[P, Q] \neq 1$.

Let $C = C_Q(P)$. Since P is A-invariant, C is A-invariant. Therefore $C = Q$ or $C = 1$. Since $[P, Q] \neq 1$, $C = 1$. In particular, $Z(T) = (Z(T) \cap P) \times (Z(T) \cap Q) \subset P$. But $Z(T)$ is an A-invariant normal subgroup of T. By our choice of P, $Z(T) = 1$. Now since D is nilpotent, $D = D_p \times D_q$ where $D_p \subseteq P$ and $D_q^x \subseteq Q$, $x \in T$. Since P and Q are abelian, D is abelian. Let $t \in C_T(A)$. Then $t^a = t$ for all $a \in A$ and therefore $t \in D$. Therefore $td^a = t^a d^a = (td)^a = (dt)^a = d^a t^a = d^a t$ for all $a \in A$ and $d \in D$. Thus $t \in Z(T) = 1$. Therefore $C_T(A) = 1$.

Let $R = AQ$, the semidirect product of Q by A. Then $Q \lhd R$, Q is a Sylow q-subgroup of R, and A is a Hall subgroup of R. Let Q^* be a Sylow q-subgroup of $C_R(P)$. Since $Q \lhd R$, $Q^* \subseteq Q$. Therefore $Q^* \subseteq C_Q(P) = 1$. Since $C_R(P) \lhd R$, we mus have $C_R(P) \subseteq A$. Let $a \in C_R(P)$, $y \in Q$, and $x \in P$. Then $(y^{-1})^a x y^a = (y^{-1}xy)^a = y^{-1}xy$. Thus $y^a y^{-1} \in C_Q(x)$. Since this is true for all $x \in P$, $y^a y^{-1} \in C_Q(P) = 1$. Since this is true for all $y \in Q$, $a \in C_R(P) \cap C_R(Q) = C_R(T)$. Since $C_A(T) = 1$, $a = 1$. Therefore R acts faithfully on P. It follows from our choice of P that R acts irreducibly on P.

Consider P as a vector space over $Z/\langle p \rangle$. By Clifford's Theorem [4, Theorem 3.4.1], $P = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ where each V_i is Q-invariant and satisfies:

(i) $V_i = X_{i1} \oplus X_{i2} \oplus \cdots \oplus X_{it}$, $1 \leqq i \leqq n$, where each X_{ij} is Q-irreducible and $X_{ij} \cong X_{rs}$ as a Q-module if and only if $i = r$; and

(ii) for $g \in AQ$, the mapping $V_i \to V_i^g$ is a permutation of $\{V_i \mid 1 \leqq i \leqq n\}$.

Let $Q_i = C_Q(X_{i1})$, $1 \leqq i \leqq n$. By (i), $Q_i = C_Q(V_i)$. Since $C_P(Q) = 1$, $Q_i \subset Q$. Thus Q/Q_i is a nontrivial abelian group represented faithfully and irreducibly on X_{i1}. Therefore Q/Q_i is cyclic [4, Theorem 3.2.2]. Since Q is elementary abelian, $|Q/Q_i| = q$. Thus Q_i is a maximal subgroup of Q. Let $v \in V_i - \{0\}$ and suppose $v^y = v$, $y \in Q^\#$. If $y \notin Q_i$, then $\langle y, Q_i \rangle = Q$ and therefore Q centralizes v. But $C_P(Q)$ is trivial. Therefore $y \in Q_i$.

Now $T = D^A$ and $D = D_p \times D_q$ where $D_p \subseteq P$ and $D_q^t \subseteq Q$, $t \in T$. Since $T = PQ$ and Q is abelian, we may assume that $t \in P$. Let $D' = D_q^t$. Then D' acts trivially on D_p. Since T is not a p-group or a q-group, $D_p \neq 1$ and $D_q \neq 1$. Assume, without loss of generality, that $D_p \subseteq V_1 \oplus V_2 \oplus \cdots \oplus V_k$ but D_p is not contained in the sum over any proper subset of $\{V_i \mid 1 \leqq i \leqq k\}$. Then for $1 \leqq j \leqq k$, there exists an element $d_j = v_1 + v_2 + \cdots + v_k \in D_p$ with $v_i \in V_i$, $1 \leqq i \leqq k$, such that $v_j \neq 0$. For $d \in D'$,

$d_j^a = d_j$. Since $v_i^d \in V_i$, $1 \leqq i \leqq k$, and the V_i's are independent, $v_j^d = v_j$. Therefore $d \in Q_j$. Since j and d were chosen arbitrarily, $D' \subseteq \bigcap_{j=1}^{k} Q_j$. In particular, if $k=n$, then $D' \subseteq \bigcap_{j=1}^{n} Q_j \subseteq C_Q(P) = 1$. Since $D' \neq 1$, $k < n$.

Let $w = w_1 + w_2 + \cdots + w_n \in P$ be chosen so that $w_i \in V_i - \{0\}$, $1 \leqq i \leqq n$. Since $T = D^A$, $w = d^a$ for some $d \in D$, $a \in A$. Since w is a p-element, d is a p-element. Therefore $d \in D_p$. Then $d = u_1 + u_2 + \cdots + u_k$ with $u_i \in V_i$, $1 \leqq i \leqq k$. Now a permutes the V_i's and $u_i^a \in V_i^a$, $1 \leqq i \leqq k$. Therefore

$$w = w_1 + w_2 + \cdots + w_n = u_1^a + u_2^a + \cdots + u_k^a \in V_1^a \oplus V_2^a \oplus \cdots \oplus V_k^a.$$

Since $w_i \neq 0$, $1 \leqq i \leqq n$, and the V_i's are independent, this implies $k=n$, a contradiction. This contradiction completes the proof.

THEOREM 3.4. *If $G = ABA$ is of type* I, *then G is solvable.*

Proof. By induction on $|G|$. Let $p \in \pi(A)$, let P be a Sylow p-subgroup of A, and let $N^* = N_G(P)$. Since G is of type I, A is abelian and therefore $A \subseteq N^*$. Therefore $N^* = A(B \cap N^*)A$. Suppose first that $N^* \subset G$. Then N^* is solvable by the induction hypothesis. By 3.2, A is a Hall subgroup of N^*. Therefore P is a Sylow p-subgroup of N^* and hence also of G. Suppose next that $N^* = G$. Let $\bar{G} = G/P$. Then $\bar{G} = \bar{A}\bar{B}\bar{A}$. Since $P \subseteq A$ and $N = N_G(A) = A$, $N_{\bar{G}}(\bar{A}) = \bar{A}$. Thus $\bar{G}$ is of type I. It follows from the induction hypothesis that $\bar{G}$ is solvable. By 3.2, $|\bar{G}|$ divides $|\bar{A}|\,|\bar{B}|^n$ for a suitable integer $n \geqq 1$. Hence p does not divide $|\bar{G}|$ and therefore P is a Sylow p-subgroup of G. Since this is true for all $p \in \pi(A)$, A is a Hall subgroup of G.

Since G is a finite group with abelian Hall subgroup A, the transfer $\tau: G \to A$ maps $A \cap Z(N)$ onto $A \cap Z(N)$ [8]. In this case, $A = A \cap Z(N)$. Therefore τ maps G onto A and $A \cap \ker(\tau) = 1$. Hence G is $\pi(A)$-separable. By 3.2, $G = AT$ where $T = (B \cap T)^A \lhd G$. By 3.3, T is nilpotent. Since A is abelian, it follows that G is solvable.

4. Groups of type II.

THEOREM 4.1. *Every ABA-group of type* II *is solvable.*

Proof. Assume false and let $G = ABA$ be a counterexample of minimal order. Suppose G possesses a nontrivial solvable normal subgroup H. Then $\bar{G} = G/H = \bar{A}\bar{B}\bar{A}$ is solvable and therefore G is solvable, a contradiction. Therefore G does not possess any nontrivial solvable normal subgroups. In particular, if $P \neq 1$ is a Sylow p-subgroup of A and if $N^* = N_G(P)$, then $N^* = A(B \cap N^*)A \subset G$. Therefore N^* is solvable. By 3.2, P is a Sylow p-subgroup of N^* and hence of G. Since this is true for all $p \in \pi(A)$, A is a Hall subgroup of G.

Case 1. $|A|$ is even. Then a Sylow 2-subgroup of G is contained in A and is therefore abelian. Furthermore, if $a \in A$ is an involution and $C = C_G(a)$, then $C = A(B \cap C)A \subset G$ and therefore C is solvable. By 2.2, G is isomorphic to a subgroup of $H^* = P\Gamma L(2, q)$ containing $H = PSL(2, q)$ where either $q = 2^n$, $n \geqq 2$, or $q \equiv 3$ or $5 \pmod 8$, $q \geqq 5$. Identifying G with its image in H^*, we have $AH =$

$A(B \cap H)A \subseteq G$. If $AH \subset G$, then AH is solvable. Since this is not the case, $AH = G$. Since $|G/H|$ divides $|A|$, $B \subseteq H$.

Suppose first that $q = 2^n$, $n \geqq 2$. By 2.3(iv), $G = HF_0$ where F_0 is of odd order and $H \cap F_0 = 1$. Thus a Sylow 2-subgroup A_2 of A is a Sylow 2-subgroup of H. By 2.3(ii) $A \subseteq C_G(A_2) = A_2$. Therefore $A = A_2 \subseteq H$ and $G = H$. By 2.3(ii), $N = N_G(A)$ is of order $q(q-1)$ and contains a cyclic subgroup of order $q-1$. Now $N = AB_0$. Therefore B_0 must be cyclic of order $q-1$. By 2.3(iii), $B \subseteq C_G(B_0) = B_0$. Therefore $A \lhd G$, a contradiction.

Suppose next that $q \equiv 3$ or $5 \pmod 8$, $q \geqq 5$. Since $PSL(2, 5) \cong PSL(2, 4)$ and $P\Gamma L(2, 5) \cong P\Gamma L(2, 4)$, we may assume that $q > 5$. Let T be a Sylow 2-subgroup of H. Replacing T by a conjugate, we may assume that $T \subseteq A$. By 2.4(vii) and 2.4(ii), $C_{H^*}(T) = T \times F$ where $|F|$ is odd. Therefore T is a Sylow 2-subgroup of G and $A \cap H = T$. Let $y \in T$ and let $C_G(y) = AB^*A$. By 2.4(iii), $|C_H(y)| = q - d$ where $d \equiv q \pmod 4$, $d = \pm 1$. Since $q > 5$, $q - d \neq 4$. Therefore $q - d$ has odd divisors and hence $B^* \neq 1$. Furthermore, since $B^* \subseteq H$, $|B^*|$ divides $q - d$. By 2.4(iv), H contains a cyclic Hall subgroup R of order $(q-d)/4$ such that R is disjoint from its conjugates. Since B^* and R are abelian, we may assume that $B^* \subseteq R$. Let $b \in B^*$. Then $\langle B, R \rangle \subseteq C_H^*(b)$ which by 2.4(v) is of order $q - d$ and is the centralizer of an involution x. By replacing R by a conjugate, we may assume that $B \subseteq R$. Since every involution of H is conjugate to one in $T \subseteq A$, $B^h \subseteq R^h \subseteq C_H(a)$ for suitable $h \in H$, $a \in T$. As above, $B_1 = B \cap C_H(a) \neq 1$. Therefore $B_1 \subseteq R \cap R^{hc}$ for a suitable element $c \in C_H(a)$. Therefore $R = R^{hc}$ and $B \subseteq R \subseteq C_H(a)$. Therefore $\langle a \rangle \lhd G$, a contradiction.

Case 2. $|A|$ is odd. Let H be a minimal subnormal subgroup of G such that $G = AH$. If $x \in G$ is of order prime to $|A|$, then $x \in H$. Therefore $B \subseteq H$ and if x is an involution, then $x \in H$. By repeated application of Theorem 1.3.8 of [4], we see that $A \cap H$ is a Hall subgroup of H.

Since G is not solvable, $|G|$ is even. Let $y = a^*ba'$ be an involution, $a^*, a' \in A$, $b \in B$. By conjugating y by a'^{-1}, we obtain an involution of the form ab, $a \in A$, $b \in B$. Since $|A|$ and $|B|$ are odd, $a \neq 1$ and $b \neq 1$. Now $ab \in H$ and $b \in H$. Therefore $a \in A \cap H$. Let $M = N_H(A \cap H)$ and let $x = a_1b_1a_2 \in M$, $a_1, a_2 \in A$, $b_1 \in B$. Then $b_1 \in H \cap N_G(A \cap H) = M$. Since ab is an involution,

$$ab = b^{-1}a^{-1}. \tag{1}$$

Conjugation of equation (1) by b_1 yields

$$a^{b_1}b = b^{-1}(a^{-1})^{b_1}. \tag{2}$$

Multiplying the inverse of equation (1) by (2) we get

$$b^{-1}a^{-1}a^{b_1}b = abb^{-1}(a^{-1})^{b_1} = a(a^{-1})^{b_1}. \tag{3}$$

Therefore $b^{-2}a^{-1}a^{b_1}b^2 = b^{-1}a(a^{-1})^{b_1}b = (b^{-1}a^{-1}a^{b_1}b)^{-1} = a^{-1}a^{b_1}$. Since $|b|$ is odd, $b^{-1}a^{-1}a^{b_1}b = a^{-1}a^{b_1}$. Combining this with equation (3) we see that $a(a^{-1})^{b_1} = a^{-1}a^{b_1}$. Since $|A|$ is odd, $a = a^{b_1}$. Therefore $a^x = a$. Since this is true for all $x \in M$,

$a \in Z(M)$. But the transfer τ of H into $A \cap H$ maps $A \cap Z(M)$ onto itself [8]. Let $K = \ker(\tau)$. Then K is subnormal in G, $G = AK$, and $|K| < |H|$. This is a contradiction to our choice of H and this contradiction completes the proof.

5. **Solvable groups of type III.** In this section we shall derive properties of solvable groups of type III which will be needed in proving our main classification theorem for all groups of type III. We first wish to make explicit several easily proved properties of groups of type III. Recall that if $G = ABA$ is of type III, then b_0 denotes an involution in B_0 which satisfies $C_A(b_0) \subseteq Z(G)$.

LEMMA 5.1. *Let $G = ABA$ be of type* III. *Let $b' \in B_0$ be an involution, and let $I_A(b') = \{a \in A \mid b'^{-1}ab' = a^{-1}\}$. Then*:

(i) *$|A|$ is odd*;

(ii) *$A = C_A(b') \times I_A(b')$; and*

(iii) *if $A \cap Z(G) = 1$, then $b_0^{-1}ab_0 = a^{-1}$ for all $a \in A$.*

Proof. Since $b_0 \in B$, $|B|$ is even. Since $(|A|, |B|) = 1$, $|A|$ is odd. Thus (i) holds.

Since $|A|$ is odd, $A = C_A(b')I_A(b')$ and $C_A(b') \cap I_A(b') = 1$ [4, Lemma 10.4.1]. Since A is abelian, $I_A(b')$ is a group and $A = C_A(b') \times I_A(b')$. Thus (ii) holds.

Statement (iii) follows immediately from (ii) and the fact that $C_A(b_0) \subseteq A \cap Z(G)$.

LEMMA 5.2. *Let $G = ABA$ be solvable of type* III. *Then*:

(i) *$G = (AB_0)T$ where $T = (B \cap T)^A$ is a nilpotent normal subgroup of G, and $(|A|, |T|) = 1$.*

(ii) *$C_T(A) \subseteq B_0 \cap AT \subseteq N \cap T \subseteq B \cap Z(G)$.*

(iii) *$B = B_0(B \cap T)$.*

(iv) *If $b^{-1}ab = a'$, $a, a' \in A$, $b \in B \cap T$, then $a = a'$.*

(v) *If $b^{-1}a^2b = a'^2$, $a, a' \in A$, $b \in B$, then $b^{-1}ab = a'$.*

(vi) *If $b' \in B_0$ is an involution and $A^* = C_A(b')$, then*

$$C_T(b') = (B \cap T)^{A^*}, \qquad C_{AT}(b') = A^*(B \cap T)^{A^*},$$

and

$$C_G(b') = (A^*(B \cap T)^{A^*})B_0 = A^*BA^*.$$

(vii) *If A acts regularly on T, then $C_G(a) \subseteq N$ for all $a \in A^{\#}$.*

Proof of (i). Since G is solvable, G is $\pi(A)$-separable. By 3.1, $G = (AB_0)T$ with $T = (B \cap T)^A \lhd G$. By 3.2, T is nilpotent. Since $T = (B \cap T)^A$ and $(|A|, |B|) = 1$, $(|A|, |T|) = 1$.

Proof of (ii). Since $T = (B \cap T)^A$, $C_T(A) \subseteq B$. Therefore $C_T(A) \subseteq B_0 \cap AT$. Since $T \lhd AT$ and $(|A|, |B|) = 1$, $B_0 \cap AT \subseteq T$. Therefore $B_0 \cap AT \subseteq N \cap T$. Now

$$[A, N \cap T] \subseteq A \cap T = 1.$$

Therefore $N \cap T \subseteq C_T(A)$. Therefore $N \cap T \subseteq C_B(A)$ and hence, since B is abelian, $N \cap T \subseteq B \cap Z(G)$.

Proof of (iii). Let $\bar{G}=G/T$. Then $\bar{G}=\bar{A}\bar{B}_0$. Since $(|A|, |B|)=1$, $\bar{B}=\bar{B}_0$. Therefore $B\subseteq B_0T$ and hence $B=B_0(B\cap T)$.

Proof of (iv). Let $b^{-1}ab=a'$, $a, a'\in A$, $b\in B\cap T$. Then $a^{-1}a'=a^{-1}b^{-1}ab\in A \cap [A, T]\subseteq A\cap T=1$. Therefore $a=a'$.

Proof of (v). Suppose $b^{-1}a^2b=a'^2$, $a, a'\in A$, $b\in B$. By (iii), $b=b_1b_2$ with $b_1\in B_0$ and $b_2\in B\cap T$. Therefore $b_2^{-1}(b_1^{-1}a^2b_1)b_2=a'^2$. Now $b_1^{-1}a^2b_1\in A$ as $b_1\in B_0$. Therefore, by (iv), $b_1^{-1}a^2b_1=a'^2\in C_G(b_2)$. Since $|a'|$ is odd, $a'\in C_G(b_2)$. Since $|A|$ is odd, each element in A has a unique square root in A. Therefore $a'=b_1^{-1}ab_1$. Therefore $b^{-1}ab=b_2^{-1}(b_1^{-1}ab_1)b_2=b_2^{-1}a'b_2=a'$.

Proof of (vi). Let $b'\in B_0$ be an involution. By 5.1(ii), $A=A^*\times A'$ where $A^*=C_A(b')$ and $A'=I_A(b')$. Let $aba_1\in C_G(b')$, $a, a_1\in A$, $b\in B$. Let $a=a^*a'$ and $a_1=a_1^*a_1'$ with a^*, $a_1^*\in A^*$ and $a', a_1'\in A'$. Then

$$a^*a'ba_1^*a_1' = aba_1 = (aba_1)^{b'} = (a^*a'ba_1^*a_1')^{b'} = a^*a'^{-1}ba_1^*a_1'^{-1}.$$

Therefore $b^{-1}a'^2b=a_1'^{-2}$. By (v), $b^{-1}a'b=a_1'^{-1}$. Hence

$$aba_1 = a^*a'ba_1^*a_1' = a^*b(b^{-1}a'b)a_1'a_1^* = a^*ba_1^*.$$

It follows immediately from the above that each element of $C_T(b')$ must be of the form $a^{*-1}ba^*$ with $a^*\in A^*$, and $b\in B\cap T$, and each element of $C_{AT}(b')$ must be of the form $a_1^*a^{*-1}ba^*$ with $a_1^*, a^*\in A^*$ and $b\in B\cap T$. On the other hand,

$$\{a^{*-1}ba^* \mid a^*\in A^* \text{ and } b\in B\cap T\} \subseteq C_T(b'),$$

and

$$\{a_1^*a^{*-1}ba^* \mid a_1^*, a^*\in A^* \text{ and } b\in B\cap T\} \subseteq C_{AT}(b').$$

Therefore $C_T(b')=(B\cap T)^{A^*}$ and $C_{AT}(b')=A^*(B\cap T)^{A^*}$. Now $C_G(b')=C_{AT}(b')B_0$. Therefore $C_G(b')=(A^*(B\cap T)^{A^*})B_0$. Since B is abelian, $B_0\subseteq N_G(A^*)$. Therefore

$$C_G(b') = (A^*(B\cap T)^{A^*})B_0 \subseteq \{a^*ba_1^* \mid a^*, a_1^*\in A^*, b\in B\} \subseteq C_G(b').$$

Therefore $C_G(b')=A^*BA^*$.

Proof of (vii). Suppose A acts regularly on T. Let $a\in A^{\#}$ and let $b\in C_B(a)$. Let $b=b_1b_2$, $b_1\in B_0$, $b_2\in B\cap T$. Then $b_2^{-1}(b_1^{-1}ab_1)b_2=a$. By (iv), $b^{-1}ab_1=a$ and $b_2\in C_B(a)$. Since A acts regularly on T, $b_2=1$. Therefore $C_B(a)\subseteq B_0$. Therefore $C_G(a)=AC_B(a)A\subseteq AB_0A=N$.

We next derive some properties of groups of type III in which $C_G(a)\subseteq N$ for all $a\in A^{\#}$. These will be used later in this section and again in §7.

PROPOSITION 5.3. *Let $G=ABA$ be of type* III. *Suppose $N\subset G$ and $C_G(a)\subseteq N$ for all $a\in A^{\#}$. Then*:

(i) $b_0^{-1}ab_0=a^{-1}$ *for all* $a\in A$.

(ii) $A\cap A^x\neq 1$ *implies* $x\in N$.

(iii) *If* $b_1ab=a_1$, $a, a_1\in A$, $a\neq 1$, $b, b_1\in B$, *then* $b_1=b^{-1}\in N$.

(iv) *If* $aba_1=a'b'a_1'$, $a, a_1, a', a_1'\in A$, $b, b'\in B$, *then either* $a=a'$, $b=b'$, *and* $a_1=a_1'$, *or* $b=b'\in N$.

(v) $|G|=|A|\,|B|(|A|-(|A|-1)/|B/B_0|)$.

(vi) *If* $b^{-1}ab=a_1b_1a_2$, $a, a_1, a_2 \in A$, $a\neq 1$, $b\in B-B_0$, $b_1 \in B$, *then* b_1 *is an involution in* $B-B_0$ *and* $a_1=a_2\neq 1$.

(vii) *If* $b^{-1}ab=a_1b_1a_1$ *and* $b^{-1}a'b=a_2b_1a_2$, $a, a', a_1, a_2 \in A^{\#}$, $b, b_1 \in B-B_0$, *then either* $a=a'$ *or* $a^{-1}=a'$.

(viii) *If G is solvable, then* $|A|=|B/B_0|+1$ *and* $b_0(B\cap Z(G))$ *is the only involution of* $B_0/(B\cap Z(G))$.

Proof of (i). Since $N\subset G$ and $C_G(a)\subseteq N$ for all $a\in A^{\#}$, $A\cap Z(G)=1$. By 5.1(iii), $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$.

Proof of (ii). Suppose $A\cap A^x\neq 1$. Then $\langle A, A^x\rangle \subseteq C_G(A\cap A^x)\subseteq N$. Since A is a normal Hall subgroup of N, $A=A^x$. Therefore $x\in N$.

Proof of (iii). Suppose $b_1ab=a_1$, $a, a_1\in A$, $a\neq 1$, $b, b_1\in B$. Then conjugation by b_0 yields $b_1a^{-1}b=a_1^{-1}$. Therefore

(1) $b^{-1}ab_1^{-1}=a_1$, and

(2) $b_1=a_1^{-1}b^{-1}a$.

Multiply equation (1) by $b_1ab=a_1$ to get $b^{-1}a^2b=a_1^2$. By (ii), $b\in N$. Therefore, using equation (2), we see that $b_1b=a_1^{-1}b^{-1}ab=a_1^{-1}a^b\in A\cap B=1$. Therefore $b_1=b^{-1}\in N$.

Proof of (iv). Suppose $aba_1=a'b'a_1'$, $a, a_1, a', a_1'\in A$, $b, b'\in B$. Then $b'^{-1}a'^{-1}ab=a_1'a_1^{-1}$. If $a'^{-1}a=1$, then since $A\cap B=1$, $b'^{-1}b=1=a_1'a_1^{-1}$.If $a'^{-1}a\neq 1$, then $b'=b\in N$ by (iii).

Proof of (v). $|G|=|N|+|\{aba' \mid a, a'\in A, b\in B-B_0\}|$. It follows from (iv) that $|\{aba' \mid a, a'\in A, b\in B-B_0\}|=|A|^2|B-B_0|$. Therefore

$$|G| = |A|\,|B_0|+|A|^2(|B|-|B_0|) = |A|\,|B|(|A|-(|A|-1)/|B/B_0|).$$

Proof of (vi). Suppose $b^{-1}ab=a_1b_1a_2$, $a, a_1, a_2\in A$, $a\neq 1$, $b\in B-B_0$, $b_1\in B$. If $b_1\in N$, then $b^{-1}ab\in N$ and, since A is a normal Hall subgroup of N, $b^{-1}ab\in A$. Then $b\in N$ by (ii). Since this is not the case, $b_1\in B-B_0$. Now

$$a_2^{-1}b_1^{-1}a_1^{-1} = b^{-1}a^{-1}b = (b^{-1}ab)^{b_0} = (a_1b_1a_2)^{b_0} = a_1^{-1}b_1a_2^{-1}.$$

By (iv), $a_1=a_2$ and $b_1^{-1}=b_1$. Since $(|A|, |B|)=1$ and $b^{-1}ab=a_1b_1a_1$, $a_1\neq 1$.

Proof of (vii). Suppose $b^{-1}ab=a_1b_1a_1$ and $b^{-1}a'b=a_2b_1a_2$, $a, a_1, a', a_2\in A^{\#}$, $b, b_1\in B-B_0$. By (vi), b_1 is an involution. Hence

$$b^{-1}aa'b = a_1b_1a_1a_2b_1a_2 = a_1b_1^{-1}a_1a_2b_1a_2.$$

Suppose $aa'\neq 1$. Then $a_1a_2\neq 1$. Now $b_1^{-1}a_1a_2b_1=a_3b_2a_4$, $a_3, a_4\in A$, $b_2\in B$. By (vi), $a_3=a_4$. Hence $b^{-1}aa'b=a_1a_3b_2a_3a_2$. By (vi), $a_1a_3=a_3a_2$ and therefore $a_1=a_2$. Therefore $b^{-1}ab=a_1b_1a_1=a_2b_1a_2=b^{-1}a'b$ and hence $a=a'$. Thus either $a=a'$, or $aa'=1$ and $a'=a^{-1}$.

Proof of (viii). Assume false and let $G=ABA$ be a counterexample of minimal order. Suppose $B\cap Z(G)\neq 1$. Let $\bar{G}=G/(B\cap Z(G))$. Then $\bar{G}=\bar{A}\bar{B}\bar{A}$ and $\bar{b}_0^{-1}\bar{a}\bar{b}_0$

$=\bar{a}^{-1}$ for all $\bar{a}\in\bar{A}$. Suppose $\bar{b}^{-1}\bar{a}\bar{b}=\bar{a}'$, $a, a'\in A$, $b\in B$. Then $b^{-1}ab=a'b'$, $b'\in B \cap Z(G)$, and hence $|b^{-1}ab|=|a'|\ |b'|$. Since $(|A|, |B|)=1$, $b'=1$. Therefore

$$N_{\bar{B}}(\bar{A}) \subseteq \bar{B}_0 \quad\text{and}\quad C_{\bar{B}}(\bar{a}) \subseteq \overline{C_B(a)}$$

for all $a\in A$. Hence $N_{\bar{G}}(\bar{A})=\bar{A}\bar{B}_0\subset\bar{G}$ and $C_{\bar{G}}(\bar{a})\subseteq N_{\bar{G}}(\bar{A})$ for all $\bar{a}\in\bar{A}$. Furthermore,

$$\bar{B}\cap Z(\bar{G}) \subseteq C_{\bar{B}}(\bar{A}) \subseteq \overline{C_B(A)} \subseteq \overline{B\cap Z(G)} = 1.$$

It follows from the minimality of G that $|A|=|\bar{A}|=|\bar{B}/\bar{B}_0|+1=|B/B_0|+1$ and $\bar{b}_0=b_0(B\cap Z(G))$ is the only involution in $\bar{B}_0=B_0/(B\cap Z(G))$. This contradicts our assumption that G is a counterexample. Therefore $B\cap Z(G)=1$.

By 5.2(i), $G=AB_0T$ with $T=(B\cap T)^A$. By 5.2(iii), $B=(B\cap T)B_0$. By 5.2(ii), $B_0\cap AT\subseteq B\cap Z(G)=1$. Let $b\in B\cap T^\#$ and $a\in A^\#$. Then $b^{-1}ab=a_1b_1a_2$, $a_1, a_2\in A$, $b_1\in B$. By (vi), $a_1=a_2$ and b_1 is an involution in $B-B_0$. Let $b_1=b_2b_3$, $b_2\in B\cap T$, $b_3\in B_0$. Then $b^{-1}ab=a_1b_2b_3a_1=b_3(b_3^{-1}a_1b_3)b_2a_1$ and therefore

$$b_3 = b^{-1}aba_1^{-1}b_2^{-1}(b_3^{-1}a_1^{-1}b_3)\in B_0\cap AT = 1.$$

Thus $b^{-1}ab=a_1b_1a_1$ with $b_1\in B\cap T^\#$. It follows from (vii) that the map $a\to b_1$ from $A^\#$ into $B\cap T^\#$ determined by $b^{-1}ab=a_1b_1a_1$ is at most two-to-one. Therefore

$$|A|-1 \leqq 2(|B/B_0|-1).$$

On the other hand, it follows from (v) that $|A|-1=n|B/B_0|$ for a suitable integer $n\geqq 1$. Therefore $n|B/B_0|\leqq 2|B/B_0|-2$ and hence $2\leqq|B/B_0|(2-n)$. Therefore $n=1$ and hence $|A|-1=|B/B_0|$.

Since G is a counterexample, there exists an involution $b'\in B_0-\{b_0\}$. Let $A^*=C_A(b')$ and $G^*=C_G(b')$. By 5.2(vi), $G^*=A^*BA^*$. By 5.1(ii), $A=A^*\times I_A(b')$. If $A^*=1$, then $b'b_0\in C_B(A)\subseteq B\cap Z(G)=1$ and therefore $b'=b_0$. Since this is not the case, $A^*\neq 1$. Since $B\cap Z(G)=1$, $G^*\subset G$.

Let $b\in N_B(A^*)$ and $a\in A^{*\#}$. Set $b=b_1b_2$, $b_1\in B_0$, $b_2\in B\cap T$. Then

$$b_2^{-1}(b_1^{-1}ab_1)b_2\in A^*.$$

By (ii), $b_2\in N$. Thus $b_2\in B_0\cap T=1$. Therefore $N_B(A^*)\subseteq B_0$. Since B is abelian, $B_0\subseteq N_B(A^*)$. Therefore $N_B(A^*)=B_0$. In particular, $N_{G^*}(A^*)\subset G^*$. Now if $b^*\in C_B(a)$, $a\in A^{*\#}$, then since $C_G(a)\subseteq N=AB_0$, $b^*\in B_0$. Thus $C_{G^*}(a)=A^*C_B(a)A^*\subseteq N_{G^*}(A^*)$ for all $a\in A^{*\#}$. Furthermore, $C_{A^*}(b_0)\subseteq C_A(b_0)=1$. By the minimality of G, $|A^*|=|B/B_0|+1$. Therefore $|A^*|=|A|$ and hence $A=A^*$. Thus $G=\langle A, B\rangle\subseteq G^*$, a contradiction. This completes the proof of (viii).

In the remainder of this section we shall treat solvable groups of type III in which $B\cap Z(G)=A\cap Z(G)=1$ and $N\subset G$. We next derive properties of the group T which arise in the decomposition of such a group described in Lemma 5.2.

Proposition 5.4. *Let T be a nontrivial group acted on by a group H. Suppose*:

(1) *$H=A\langle b_0\rangle$ with A abelian of odd order prime to $|T|$ and b_0 an involution satisfying $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$*;

(2) *T possesses a nilpotent subgroup $D \subseteq C_T(b_0)$ such that $T = D^A$; and*

(3) $C_T(A) = 1$.

Then the following conditions hold.

(i) *$C_T(b_0) = D$ and $D^a \cap D \subseteq C_D(a)$ for all $a \in A$.*

(ii) *T is a 2-group.*

(iii) *If T is abelian, $C_A(T) = 1$, and T is A-indecomposable, then A acts regularly on T.*

(iv) *If A acts regularly on T, then $|T^\#| = |D^\#|\ |A|$, T is elementary abelian, and A acts irreducibly on T.*

(v) *If D is abelian, then T is an elementary abelian 2-group.*

Proof of (i). Suppose $d^a \in C_T(b_0)$, $d \in D$, $a \in A$. Then $d^a = (d^a)^{b_0} = (d^{b_0})^{a^{-1}} = d^{a^{-1}}$. Therefore $d^{a^2} = d$. Since $|a|$ is odd, $d^a = d$. Therefore, since $T = D^A$, $C_T(b_0) \subseteq D$. Since $D \subseteq C_T(b_0)$, $C_T(b_0) = D$. Furthermore, since $D^a \cap D \subseteq D \subseteq C_T(b_0)$, $D^a \cap D \subseteq C_D(a)$ for all $a \in A$.

Proof of (ii). By 3.3, T is nilpotent. Let $R = O(T)$. Then R char T and therefore R is H-invariant. Since $|R|$ is odd, $R = C_R(b_0)I$ where $I = \{r \in R \mid r^{b_0} = r^{-1}\}$. Let $r = d^a \in I$, $d \in D$, $a \in A$. Then $(d^{-1})^a = (d^a)^{-1} = r^{-1} = r^{b_0} = (d^a)^{b_0} = (d^{b_0})^{a^{-1}} = d^{a^{-1}}$. Therefore $(d^{-1})^{a^2} = d$. By (i), $d = d^{-1}$. Therefore $r^2 = 1$. Since $|R|$ is odd, $r = 1$. Therefore $R = C_R(b_0) \subseteq C_T(b_0) = D$. Since R is A-invariant and $D^a \cap D \subseteq C_D(a)$ for all $a \in A$, $R \subseteq C_D(A) \subseteq C_T(A) = 1$. Therefore T is a 2-group.

Proof of (iii). Suppose T is abelian, $C_A(T) = 1$, and T is A-indecomposable. Let $a \in A$ and assume $C_T(a) \neq 1$. Since $(|\langle a \rangle|, |T|) = 1$, $T = C_T(\langle a \rangle) \times [T, \langle a \rangle]$ [4, Theorem 5.2.3]. Since A is abelian, $C_T(\langle a \rangle)$ and $[T, \langle a \rangle]$ are A-invariant. Since T is A-indecomposable and $C_T(\langle a \rangle) \neq 1$, $T = C_T(\langle a \rangle)$. Thus $a \in C_A(T) = 1$. Therefore A acts regularly on T.

Proof of (iv). Assume that A acts regularly on T. If $D^{a'} \cap D^{a^*} \neq 1$, $a', a^* \in A$, then $D^a \cap D \neq 1$ where $a = a'a^{*-1}$. By (i), $C_D(a) \neq 1$. Since A acts regularly on T, $a = 1$ and hence $a' = a^*$. Therefore $|T^\#| = |(D^\#)^A| = |D^\#|\ |A|$.

Let $T^* \neq 1$ be an A-invariant subgroup of T. Then $T^* = (T^* \cap D)^A$ and therefore T^* is H-invariant. Furthermore, A acts regularly on T^*. As above, $|T^{*\#}| = |T^* \cap D^\#|\ |A|$. Thus

$$|A| = \frac{|T|-1}{|D|-1} = \frac{|T^*|-1}{|T^* \cap D|-1},$$

and hence

$$|T|\ |T^* \cap D| - |T^* \cap D| - |T| = |T^*|\ |D| - |D| - |T^*|.$$

Let $|T^* \cap D| = 2^m$, $|D| = 2^{m+r}$, $|T^*| = 2^{m+s}$, and $|T| = 2^{m+s+t}$. Then

$$2^{2m+s+t} - 2^m - 2^{m+s+t} = 2^{2m+s+r} - 2^{m+r} - 2^{m+s}.$$

Therefore $2^{m+s+t} - 1 - 2^{s+t} = 2^{m+s+r} - 2^r - 2^s$. This is only possible if $2^s = 1$ or $2^r = 1$. If $2^s = 1$, then $|T^*| = |T^* \cap D|$ and therefore $T^* \subseteq D$. Then $T^* \subseteq D^a \cap D$ for all

$a \in A$ and, by (i), $T^* \subseteq C_T(A) = 1$. Since this is not the case, $2^r = 1$. Hence $D \subseteq T^*$ and therefore $T = T^*$. Since T^* was an arbitrary A-invariant subgroup of T, $T = \Omega_1(Z(T))$ is elementary abelian and A acts irreducibly on T.

Proof of (v). Assume false and let $T = D^A$ with D abelian be a counterexample of minimal order. Let $A^* = C_A(T)$. Then $A^* \lhd H$. By replacing H by H/A^*, we may assume $C_A(T) = 1$. By (ii), T is a 2-group.

Let S be an A-invariant subgroup of T. Then $S = (S \cap D)^A$ and therefore S is H-invariant. Furthermore, $S \cap D \subseteq C_S(b_0)$ and $C_S(A) \subseteq C_T(A) = 1$. Suppose further that $S \lhd T$. Let $\tilde{T} = T/S$. Then H acts on $\tilde{T}$, $\tilde{T} = \tilde{D}^A$, and $\tilde{D} \subseteq C_{\tilde{T}}(b_0)$. It follows from the minimality of T that any A-invariant proper subgroup of T is elementary abelian. Furthermore, if S is a nontrivial, A-invariant, normal subgroup of T, and if $C_{T/S}(A) = 1$, then T/S is elementary abelian.

Suppose T possesses two disjoint, nontrivial, A-invariant, normal subgroups M_1 and M_2. Let $C_{T/M_i}(A) = R_i/M_i$, $i = 1, 2$. Then $R_i/M_i \lhd T/M_i$, $R_i \lhd T$, and R_i is A-invariant, $i = 1, 2$. Let $x \in R_1 \cap R_2$ and $a \in A$. Then $x^a = xm_1 = xm_2$, $m_1 \in M_1$, $m_2 \in M_2$. Since $M_1 \cap M_2 = 1$, $x^a = x$. Hence $R_1 \cap R_2 \subseteq C_T(A) = 1$. Let $C_{T/R_i}(A) = S_i/R_i$, $i = 1, 2$. Then A centralizes S_i/M_i [4, Theorem 5.3.2] and therefore $S_i \subseteq R_i$, $i = 1, 2$. Therefore $C_{T/R_i}(A) = 1$, $i = 1, 2$. Hence T/R_1, T/R_2, and therefore $L = T/R_1 \times T/R_2$, are elementary abelian. But $\phi(x) = (xR_1, xR_2)$ defines an embedding of T into L. Hence T is elementary abelian, a contradiction. Therefore T does not possess two such A-invariant subgroups. In particular A acts indecomposably on T and on $Z(T)$. It follows immediately from (iii) and (iv) that T is not abelian.

Let $Z = Z(T)$ and $A_0 = C_A(Z)$. Since Z is A-indecomposable and abelian, A/A_0 acts regularly on Z by (iii) (applied with H replaced by H/A_0). By (iv), Z is elementary abelian and

$$|Z^\#| = |(Z \cap D^\#)^A| = |Z \cap D^\#|\,|A/A_0|. \tag{1}$$

Let $\bar{T} = T/Z$ and let $C_{\bar{T}}(A) = S/Z$. Then S is A-invariant. Let $s \in S$. Then $s = d^a$, $d \in S \cap D$, $a \in A$. Since $d \in S$, $d^a = dz$, $z \in Z$. Therefore $s = dz$. If $s' \in S$, then $s' = d'z'$, $d' \in S \cap D$, $z' \in Z$, and $ss' = dzd'z' = d'z'dz = s's$ since $z, z' \in Z$ and D is abelian. Thus S is abelian and hence $S \subset T$. Therefore S is elementary abelian. Since $(|A|, |S|) = 1$, $S = Z \times S'$ with S' an A-invariant subgroup of S. Now for $a \in A$ and $s \in S'$, $s^a = sz$, $z \in Z$. Since S' is A-invariant and $S' \cap Z = 1$, $s^a = s$. Thus $S' \subseteq C_T(A) = 1$ and therefore $C_{\bar{T}}(A) = 1$. Therefore $\bar{T}$ is elementary abelian. Hence $1 \neq [T, T] \subseteq \phi(T) \subseteq Z$. Now Z is elementary abelian and A-indecomposable and $(|A|, |Z|) = 1$. Therefore Z is A-irreducible. Hence $[T, T] = \phi(T) = Z$ and therefore Z is a special 2-group.

Since $(|A|, |\bar{T}|) = 1$, we can express $\bar{T}$ as $\bar{T} = \bar{T}_1 \times \bar{T}_2 \times \cdots \times \bar{T}_n$ where each $\bar{T}_i$ is A-irreducible, $1 \leqq i \leqq n$. Let T_i be the full inverse image of $\bar{T}_i$ in T, $1 \leqq i \leqq n$. Then T_i is A-invariant, $1 \leqq i \leqq n$. If $n \geqq 3$, then $T_iT_j \subset T$ and therefore T_iT_j is abelian, $1 \leqq i$, $\leqq n$. Then $T = T_1T_2 \cdots T_n$ is abelian. Since this is not the case, $n \leqq 2$.

Suppose $n=1$. That is, A acts irreducibly on $\bar{T}$. If $a \in C_A(\bar{T})$, then since $\bar{T}=T/\phi(T)$, $a \in C_A(T)$ [4, Theorem 5.1.4]. Therefore $C_A(\bar{T}) \subseteq C_A(T)=1$. By (iii), A acts regularly on $\bar{T}$. Therefore, by (iv),

$$(2) \qquad |\bar{T}^{\#}| = |\bar{D}^{\#}|\,|A|.$$

Let $E=D-(Z \cap D)$. Then $T^{\#}=(D^{\#})^A=Z^{\#} \cup E^A$ and the union is disjoint. Therefore

$$(3) \qquad |T^{\#}| = |Z^{\#}|+|E^A|.$$

Suppose $E^{a'} \cap E^{a^*} \neq 1$, $a', a^* \in A$. Then $E^a \cap E \neq 1$, $a=a'a^{*-1}$. By (i), $C_E(a) \neq 1$ and therefore $C_{\bar{T}}(a) \neq 1$. Since A acts regularly on $\bar{T}$, $a=1$ and hence $a'=a^*$. Therefore

$$(4) \qquad |E^A| = |E|\,|A|.$$

By combining equations (3) and (4), we see that

$$\begin{aligned}|T| &= |Z|+|E|\,|A| = |Z|+(|D|-|Z \cap D|)|A| = |Z|+|Z \cap D|\left(\frac{|D|}{|Z \cap D|}-1\right)|A| \\ &= |Z|+|Z \cap D|\,|\bar{D}^{\#}|\,|A|.\end{aligned}$$

By combining this with equation (2), we see that

$$(5) \qquad |T|-|Z| = |Z \cap D|(|\bar{T}|-1).$$

Since T is a 2-group, $|\bar{T}|-1$ is odd. Since $|Z|$ is a power of 2 and divides the left-hand side of equation (5), $|Z|$ divides $|Z \cap D|$. Therefore $Z=Z \cap D$. Hence $Z \subseteq D^a \cap D$ for all $a \in A$. By (i), $Z \subseteq C_T(A)=1$, a contradiction. Therefore $n=2$.

Thus $\bar{T}=\bar{T}_1 \times \bar{T}_2$ where $\bar{T}_1$ and $\bar{T}_2$ are nontrivial and A-irreducible. Now $T_1 \subset T$ and $T_2 \subset T$. Hence T_1 and T_2 are elementary abelian. Since $(|A|, |T_i|)=1$, $T_i=Z \times U_i$ with U_i A-irreducible, $i=1, 2$. Furthermore,

$$U_i \cap T_j \subseteq U_i \cap T_i \cap T_j \subseteq U_i \cap Z=1, \qquad 1 \leqq i,j \leqq 2, i \neq j.$$

Let $A_i=C_A(\bar{T}_i)$, $i=1, 2$. Then $A_i=C_A(U_i) \subset A$ and A/A_i acts faithfully, irreducibly, and therefore, by (iii), regularly on U_i, $i=1, 2$. Let $D_i=U_i \cap D$, $i=1, 2$. By (iv),

$$(6) \qquad |U_i^{\#}| = |(D_i^{\#})^A| = |D_i^{\#}|\,|A/A_i|, \qquad i = 1, 2.$$

Since $\bar{T}=T/\phi(T)$, $C_A(\bar{T}) \subseteq C_A(T)=1$. Therefore $A_1 \cap A_2=1$. Suppose that $a \in A_0 \cap A_1$. Let $u_1 \in U_1$ and $u_2 \in U_2$. Then $u_2^{-1}u_1^{-1}u_2u_1=z \in Z$. Therefore $u_2^{-1}u_1^{-1}u_2u_1 =z=z^a=(u_2^{-1})^a u_1^{-1}u_2^a u_1$ and hence $u_2(u_2^{-1})^a \in C_{U_2}(u_1)$. Since this is true for all $u_1 \in U_1$, $u_2(u_2^{-1})^a \in C_{U_2}(U_1)=C_{U_2}(T_1)$. Now $C=C_T(T_1)$ is A-invariant and contains T_1. Therefore, either $\bar{C}=\bar{T}_1$ or $\bar{C}=\bar{T}$. If $\bar{C}=\bar{T}$, then $T_1 \subseteq Z$ and therefore $\bar{T}_1=1$. Therefore $\bar{C}=\bar{T}_1$ and hence $C=T_1$. Therefore $u_2(u_2^{-1})^a \in U_2 \cap T_1=1$. Since this is true for all $u_2 \in U_2$, $a \in A_1 \cap A_2=1$. Therefore $A_0 \cap A_1=1$. Similarly $A_0 \cap A_2=1$.

Suppose $zu_1u_2=z'u_1'u_2'$, $z, z' \in Z$, $u_i, u_i' \in U_i$, $i=1, 2$. Then $u_1'^{-1}z'^{-1}zu_1=u_2'u_2^{-1} \in T_1 \cap U_2=1$. Therefore $u_2'=u_2$. Similarly $u_1'=u_1$ and therefore $z=z'$. Since $T=T_1T_2 =ZU_1U_2$, we conclude that every element of T has a unique representation in the form zu_1u_2, $z \in Z$, $u_i \in U_i$, $i=1, 2$. Now let $d \in D$ and suppose $d=zu_1u_2$, $z \in Z$, $u_1 \in U_1$, $u_2 \in U_2$. Then $z^{b_0}u_1^{b_0}u_2^{b_0}=d^{b_0}=d=zu_1u_2$. Since Z, U_1, and U_2 are A-invariant, they are also b_0-invariant. Therefore $z^{b_0}=z$, $u_1^{b_0}=u_1$, and $u_2^{b_0}=u_2$. Thus $z, u_1, u_2 \in C_T(b_0)=D$. Therefore $D=(Z \cap D)D_1D_2$.

Let

$$X_i = \{(dd_i)^a \mid d \in Z \cap D^{\#}, d_i \in D_i^{\#}, a \in A\}, \qquad i = 1, 2;$$

$$X_3 = \{(d_1d_2)^a \mid d_1 \in D_1^{\#}, d_2 \in D_2^{\#}, a \in A\};$$

and

$$X_4 = \{(dd_1d_2)^a \mid d \in Z \cap D^{\#}, d_1 \in D_1^{\#}, d_2 \in D_2^{\#}, a \in A\}.$$

Then

$$(7) \qquad |T_i^{\#}| = |(T_i \cap D^{\#})^A| = |Z^{\#}|+|U_i^{\#}|+|X_i|, \qquad i = 1, 2,$$

and

$$(8) \qquad |T^{\#}| = |(D^{\#})^A| = |Z^{\#}|+|U_1^{\#}|+|U_2^{\#}|+|X_1|+|X_2|+|X_3|+|X_4|.$$

Suppose $(dd_1d_2)^{a'}=(d'd_1'd_2')^{a^*}$, $a', a^* \in A$, $d, d' \in Z \cap D$, $d_i, d_i' \in D_i$, $i=1, 2$. Then $d^ad_1^ad_2^a=d'd_1'd_2'$, $a=a'a^{*-1}$. Hence $d^a=d'$, $d_1^a=d_1'$, and $d_2^a=d_2'$. By (i), $d^a=d$, $d_1^a=d_1$, and $d_2^a=d_2$. Now A/A_0 acts regularly on Z, A/A_i acts regularly on U_i, $i=1, 2$, and $A_0 \cap A_1 = A_0 \cap A_2=A_1 \cap A_2=1$. Therefore either $a=1$ and $a'=a^*$, or two of d, d_1, and d_2 are trivial. Therefore

$$(9) \qquad |X_i| = |Z \cap D^{\#}|\,|D_i^{\#}|\,|A|, \qquad i = 1, 2;$$

$$(10) \qquad |X_3| = |D_1^{\#}|\,|D_2^{\#}|\,|A|;$$

and

$$(11) \qquad |X_4| = |Z \cap D^{\#}|\,|D_1^{\#}|\,|D_2^{\#}|\,|A|.$$

By combining equations (7) and (9), we see that

$$|Z|\,|U_i|-1 = |T_i^{\#}| = |Z^{\#}|+|U_i^{\#}|+|Z \cap D^{\#}|\,|D_i^{\#}|\,|A|, \qquad i = 1, 2.$$

By combining this with equations (1) and (6), we see that

$$|Z|\,|U_i| = |Z|+|U_i^{\#}|+|Z^{\#}|\,|U_i^{\#}|(|A_0|\,|A_i|/|A|), \qquad i = 1, 2.$$

Therefore $|Z^{\#}|\,|U_i^{\#}|=|Z|\,|U_i|-|Z|-|U_i^{\#}|=|Z^{\#}|\,|U_i^{\#}|(|A_0|\,|A_i|/|A|)$ and hence

$$(12) \qquad |A_0|\,|A_i|/|A| = 1, \qquad i = 1, 2.$$

By combining equations (8) through (11), we see that

$$|T^{\#}| = |Z^{\#}|+|U_1^{\#}|+|U_2^{\#}|+|Z \cap D^{\#}|(|D_1^{\#}|+|D_2^{\#}|)|A|+|Z \cap D|\,|D_1^{\#}|\,|D_2^{\#}|\,|A|.$$

Together with equations (1), (6), and (12), this implies

$$\begin{aligned}|T| &= |Z|+|U_1^\#|+|U_2^\#|+|Z^\#|(|U_1^\#|+|U_2^\#|)+|Z\cap D|\;|D_1^\#|\;|D_2^\#|\;|A|\\ &= |Z|+|Z|(|U_1^\#|+|U_2^\#|)+|Z\cap D|\;|D_1^\#|\;|D_2^\#|\;|A|.\end{aligned}$$

Therefore, since $|Z|$ divides $|T|$, $|Z|$ must also divide $|Z\cap D|\;|D_1^\#|\;|D_2^\#|\;|A|$. Since T is a 2-group and $T_i=D_i^A\neq 1$, $i=1,2$, $|D_1^\#|$ and $|D_2^\#|$ are odd whereas $|Z|$ is a power of 2. Hence, since $|A|$ is odd, $|Z|$ divides $|Z\cap D|$. Therefore $Z\subseteq D$. Since Z is A-invariant, $Z\subseteq D^a\cap D$ for all $a\in A$. By (i), $Z\subseteq C_T(A)=1$, a contradiction. This contradiction completes the proof of (v).

Let $G=ABA$ be a solvable group of type III in which $A\cap Z(G)=B\cap Z(G)=1$ and $N\subset G$. By 5.2(i), $G=(AB_0)T$ where $T=(B\cap T)^A$ is a nilpotent normal subgroup of G, and $(|A|,|T|)=1$. Since $N\subset G$, $T\neq 1$. By 5.1(i) and (iii), $|A|$ is odd and $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$. Since $B\cap Z(G)=1$, it follows from 5.2(iii) that $C_T(A)=B_0\cap AT=N\cap T=1$. Thus $H=A\langle b_0\rangle$ and T satisfy the conditions and hence the conclusions of Proposition 5.4. In particular, since $B\cap T$ is abelian, T is an elementary abelian 2-group. Since $(|A|,|T|)=1$, T and any A-invariant subgroups of T are completely A-reducible. Note that if T^* is an A-invariant subgroup of T, then $T^*=(B\cap T^*)^A$ and therefore T^* is B_0-invariant. In the remainder of this section we shall obtain further information about G.

PROPOSITION 5.5. *Let T^* be an irreducible A-invariant subgroup of T and $A^*=C_A(T^*)$. Let B_2 be a Sylow 2-subgroup of B_0 and $B^*=\Omega_1(B_2)$. Then*

(i) *B^* acts on A/A^*,*

(ii) *$|B^*:C_{B^*}(A/A^*)|=2$, and*

(iii) *$|T^*|=|B\cap T^*|^2$.*

Proof. Since T^* is A-invariant, it is also B_0-invariant. Hence $A^*=C_A(T^*)$ is B_0-invariant. Since $B^*\subseteq B_0$, B^* acts on A and on A^*. Hence B^* acts on A/A^*. Thus (i) holds.

Let $K=(AT^*)B^*$ and let $B'=(B\cap T^*)B^*$. Then $K=AB'A$, $A^*\lhd K$, and $b_0\in B'$. Since $B_0\cap AT^*\subseteq B\cap Z(G)=1$, $N_{B'}(A)=B^*$ and $B'=(B\cap T^*)\times B^*$.

Let $\bar K=K/A^*$. Then $\bar K=\bar A\bar B'\bar A=\bar A\bar T^*\bar B^*$, $\bar b_0^{-1}\bar a\bar b_0=\bar a^{-1}$ for all $\bar a\in\bar A$,

$$\bar B'=\overline{(B\cap T^*)}\times\bar B^*,\quad N_{\bar B'}(\bar A)=\overline{N_{B'}(A)}=\bar B^*,\quad\text{and}\quad \bar T^*=\overline{(B\cap T^*)}^{\bar A}.$$

In particular, $\bar K$ is of type III. Since $(|A|,|BT|)=1$, $A^*\cap T^*\subseteq A^*\cap B'T^*=1$. Hence

$$\overline{B\cap T^*}=\bar B'\cap\bar T^*,\quad \bar T^*\neq 1,$$

$\bar A$ acts irreducibly on $\bar T^*$,

$$C_{\bar A}(\bar T^*)=\overline{C_A(T^*)}=\bar A^*=1,\quad\text{and}\quad C_{\bar T^*}(\bar A)=\overline{C_{T^*}(A)}=1.$$

In particular, $\bar T^*=(\bar B'\cap\bar T^*)^{\bar A}$ and $N_{\bar K}(\bar A)\subset\bar K$.

By 5.4(iii), $\bar A$ acts regularly on $\bar T^*$. By 5.4(iv),

$$|\bar T^*|-1=(|\overline{B\cap T^*}|-1)|\bar A|. \tag{1}$$

By 5.2(vii), $C_{\bar{K}}(\bar{a}) \subseteq N_{\bar{K}}(\bar{A})$ for all $\bar{a} \in \bar{A}^{\#}$. Since

$$N_{\bar{B}'}(\bar{A}) = \bar{B}^* \quad \text{and} \quad \bar{B}' = (\overline{B \cap T^*}) \times \bar{B}^*,$$

it follows from 5.3(viii) that

$$|\bar{A}| = |\overline{B \cap T^*}| + 1, \tag{2}$$

and

(3) $\bar{b}_0$ is the only involution in $\bar{B}^*/(\bar{B}' \cap Z(\bar{K}))$.

Since B^* is an elementary abelian 2-group, (3) implies that $|\bar{B}^* : \bar{B}' \cap Z(\bar{K})| = 2$. Since $\bar{B}' \cap Z(\bar{K}) \subseteq N_{\bar{B}'}(\bar{A}) = \bar{B}^*$ and $\bar{B}'$ is abelian,

$$\bar{B}' \cap Z(\bar{K}) = C_{\bar{B}^*}(\bar{A}) = \overline{C_{B^*}(A/A^*)}.$$

Therefore

$$|B^* : C_{B^*}(A/A^*)| = |\bar{B}^* : \overline{C_{B^*}(A/A^*)}| = |\bar{B}^* : \bar{B}' \cap Z(\bar{K})| = 2.$$

Hence (ii) holds.

By combining equations (1) and (2), we see that

$$|\bar{T}^*| = (|\overline{B \cap T^*}| - 1)(|\overline{B \cap T^*}| + 1) + 1 = |\overline{B \cap T^*}|^2.$$

Since $A \cap T = 1$,

$$|\bar{T}^*| = |T^*| \quad \text{and} \quad |\overline{B \cap T^*}| = |B \cap T^*|.$$

Therefore $|T^*| = |B \cap T^*|^2$. Thus (iii) holds.

COROLLARY 5.6. $|T| = |B \cap T|^2$.

Proof. Let $T = T_1 \times T_2 \times \cdots \times T_n$ where each T_i is A-irreducible, $1 \leqq i \leqq n$. Let $b \in B \cap T$. Then $b = t_1 t_2 \cdots t_n$, $t_i \in T_i$, $1 \leqq i \leqq n$. Hence $t_1 t_2 \cdots t_n = b = b^{b_0} = t_1^{b_0} t_2^{b_0} \cdots t_n^{b_0}$. Since the T_i's are b_0-invariant and independent, $t_i^{b_0} = t_i$, $1 \leqq i \leqq n$. Hence, by 5.4(i), $t_i \in B \cap T_i$, $1 \leqq i \leqq n$. Therefore $B \cap T = (B \cap T_1) \times (B \cap T_2) \times \cdots \times (B \cap T_n)$. By 5.5(iii), $|T_i| = |B \cap T_i|^2$, $1 \leqq i \leqq n$. Therefore

$$|T| = |T_1|\,|T_2| \cdots |T_n| = |B \cap T_1|^2 |B \cap T_2|^2 \cdots |B \cap T_n|^2 = |B \cap T|^2.$$

PROPOSITION 5.7. *Let $b \in B_0 - \{b_0\}$ be an involution, $A_1 = C_A(b)$, and $A_2 = C_A(bb_0)$. Then $AT = A_1 T_1 \times A_2 T_2$ where T_1 and T_2 are A-invariant subgroups of T.*

Proof. Since $b_0^{-1} a b_0 = a^{-1}$ for all $a \in A$, $A_2 = \{a \in A \mid b^{-1} a b = a^{-1}\} = I_A(b)$. By 5.1(ii), $A = A_1 \times A_2$. If $A_1 = 1$, then $bb_0 \in C_B(A) \subseteq B \cap Z(G) = 1$ and therefore $b = b_0$. Since this is not the case, $A_1 \neq 1$. Since T is an elementary abelian 2-group and $(|A|, |T|) = 1$, $T = T_1 \times T_2$ where $T_1 = [A_1, T]$ and $T_2 = C_T(A_1)$ [4, Theorem 5.2.3]. Since A is abelian, both T_1 and T_2 are A-invariant. Furthermore, $C_{T_1}(A_1) = 1$. If $T_1 = 1$, then $T = T_2$ and $AT = A_1 \times A_2 T$ satisfies the conclusion of the proposition. We may therefore assume that $T_1 \neq 1$.

Let T^* be an A-irreducible constituent of T_1 and let $A^* = C_A(T^*)$. Let B^* be defined as in 5.5 and $B_1 = C_{B^*}(A/A^*)$. Then $|B^*/B_1| = 2$. Suppose $b \notin B_1$. Then

$b_0 \in B^* = \langle b, B_1 \rangle \subseteq C_{B^*}(A_1A^*/A^*)$. Since b_0 inverts the elements of A, this implies that $A_1 \subseteq A^* = C_A(T^*)$. This is not the case since $C_{T_1}(A_1) = 1$. Therefore $b \in B_1$. In particular, b centralizes A_2A^*/A^*. Since b inverts the elements of A_2, this implies that $A_2 \subseteq A^* = C_A(T^*)$. Since this is true for all A-irreducible constituents of T_1, $A_2 \subseteq C_A(T_1)$. Therefore $AT = A_1T_1 \times A_2T_2$.

PROPOSITION 5.8. $[A, O(B_0)] \subseteq C_A(T)$.

Proof. Since T is completely A-reducible, it suffices to show that $[A, O(B_0)] \subseteq C_A(T^*)$ for any A-irreducible constituent T^* of T. Let $H = (AB_0)T^*$. Then

$$H = A(B_0(B \cap T^*))A, \quad A \cap Z(H) \subseteq C_A(b_0) = 1, \quad B \cap Z(H) \subseteq C_B(A) = 1,$$

and, since $T^* \neq 1$ and $T^* \cap B_0 \subseteq B \cap Z(G) = 1$, $N_H(A) \subset H$. Thus we may assume that $H = G$ and T is A-irreducible.

We consider T as a vector space over $F = Z/\langle 2 \rangle$. By 5.6, $|T| = |B \cap T|^2$. Therefore $\dim_F (C_T(b)) \geqq (1/2) \dim_F (T)$ for all $b \in B_0$. Let L be an algebraic closure of F and let $T_L = T \otimes_F L$. Then each element $g \in N$ induces a linear transformation of T_L, the characteristic roots of g on T and on T_L are the same, and

$$\dim_L (C_{T_L}(b)) \geqq (1/2) \dim_L (T_L)$$

for all $b \in B_0$ (see §3.1 of [4]).

Let $B' = O(B_0)$. Since $|AB'|$ is odd and $\operatorname{char}(L) = 2$, we can express T_L as $T_L = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ where AB' acts irreducibly on each V_i, $1 \leqq i \leqq m$. Let j be chosen so that $1 \leqq j \leqq m$. By Clifford's Theorem, $V_j = U_1 \oplus U_2 \oplus \cdots \oplus U_s$ where each U_i is A-invariant and satisfies:

(1) $U_i = X_{i1} \oplus X_{i2} \oplus \cdots \oplus X_{it}$, $1 \leqq i \leqq s$, where each X_{ik} is A-irreducible and $X_{ik} \cong X_{ru}$ as an A-module if and only if $i = r$; and

(2) for $g \in AB'$, the mapping $U_i \to U_i^g$ is a permutation of $\{U_i \mid 1 \leqq i \leqq s\}$.

Assume that $[A, B'] \nsubseteq C_A(T)$ and let $b \in B'$ be such that $[A, \langle b \rangle] \nsubseteq C_A(T)$. Suppose $U_i = U_i^b$ for some i, $1 \leqq i \leqq s$. Since A acts irreducibly on X_{i1}, L is algebraically closed, and $(\operatorname{char}(L), |A|) = 1$, each element $a \in A$ acts as a scalar on X_{i1} [4, Theorem 3.2.5]. Since $X_{ik} \cong X_{i1}$ as an A-module, $1 \leqq k \leqq t$, each element $a \in A$ acts as a scalar on U_i. Thus for $a \in A$, $b^{-1}a^{-1}ba \in C_N(U_i)$. Therefore 1 is a characteristic root of $b^{-1}a^{-1}ba$ on T_L and hence on T. Therefore $C_T(b^{-1}a^{-1}ba) \neq 1$. But $b^{-1}a^{-1}ba \in A$, A is abelian, and T is A-irreducible. Therefore $C_T(b^{-1}a^{-1}ba) = T$. Since this is true for all $a \in A$, $[A, \langle b \rangle] \subseteq C_A(T)$, a contradiction. Therefore b acts as a fixed point free permutation of $\{U_n \mid 1 \leqq i \leqq s\}$.

By rearranging the order of the U_i's, we may assume that

$$(U_1, U_2, \ldots, U_{n_1}), (U_{n_1+1}, U_{n_1+2}, \ldots, U_{n_2}), \ldots, (U_{n_r+1}, U_{n_r+2}, \ldots, U_s)$$

are the cycles of b. Then

$$\dim_L (C_{V_j}(b)) = \dim_L (U_1) + \dim_L (U_{n_1+1}) + \cdots + \dim_L (U_{n_r+1}).$$

Furthermore, since the length of each cycle divides $|b|$,

$$\dim_L(V_j) \geqq p(\dim_L(U_1)+\dim_L(U_{n_1+1})+\cdots+\dim_L(U_{n_r+1}))$$

where p is the smallest prime dividing $|b|$. Therefore

$$(1/p)\dim_L(V_j) \geqq \dim_L(C_{V_j}(b)).$$

Let $w_k=\dim_L(V_k)$ and $u_k=\dim_L(C_{V_k}(b))$, $1\leqq k\leqq m$. Since j was chosen arbitrarily and $|b|$ is odd, $u_k\leqq(1/p)w_k<(1/2)w_k$, $1\leqq k\leqq m$. Since each V_k is b-invariant, $1\leqq k\leqq m$, $C_{T_L}(b)=C_{V_1}(b)\oplus C_{V_2}(b)\oplus\cdots\oplus C_{V_m}(b)$. Therefore

$$\dim_L(C_{T_L}(b)) = \sum_{k=1}^{m} u_k < (1/2)\sum_{k=1}^{m} w_k = (1/2)\dim_L(T_L),$$

a contradiction. Therefore $[A, B']\subseteq C_A(T)$.

COROLLARY 5.9. *Let* $A_1=C_A(O(B_0))$ *and* $A_2=[A, O(B_0)]$. *Then the following conditions hold.*

(i) $A=A_1\times A_2$ *and* $A_1\neq 1$.

(ii) $G=(A_1T\times A_2)B_0$.

(iii) $C_G(O(B_0))=A_1TB_0=A_1BA_1$.

Proof. Since A is abelian and $(|A|, |B|)=1$, $A=A_1\times A_2$ [4, Theorem 5.2.3]. By 5.8, $A_2\subseteq C_A(T)$. Therefore $G=(A_1T\times A_2)B_0$ and $T=(B\cap T)^{A_1}\subseteq C_G(O(B_0))$. If $A_1=1$, then $N_G(A)=N_G(A_2)\supseteq T$ and therefore $A\lhd G$. Since this is not the case, $A_1\neq 1$. Finally, since $C_{A_2}(O(B_0))=1$, $C_{AT}(O(B_0))=A_1T$. Hence

$$C_G(O(B_0)) = C_{AT}(O(B_0))B_0 = A_1TB_0 = A_1(B\cap T)^{A_1}B_0 = A_1BA_1.$$

6. **Factorizable ABA-groups.** If $G=ABA$ and $G=A_1B_1A_1\times A_2B_2A_2\times\cdots\times A_nB_nA_n$ where

(1) $A_i\subseteq A$ and $B_i\subseteq B$, $1\leqq i\leqq n$,

(2) $A_1B_1A_1$ is either trivial or solvable, and

(3) either $A_iB_iA_i=1$, $2\leqq i\leqq n$, or $A_iB_iA_i\simeq PSL(2, 2^{m_i})$, $m_i\geqq 2$, $2\leqq i\leqq n$, then we shall say that G is *factorizable*. We shall refer to the subgroups $G_i=A_iB_iA_i$ as *components of* G. We shall often be dealing with groups $G=ABA$ which we shall know to be factorizable. If in such a situation we write $G=A_1B_1A_1\times A_2B_2A_2\times\cdots\times A_nB_nA_n$, then it is to be understood that (1), (2), and (3) hold. Furthermore, if $G_i=A_iB_iA_i$ is a component of G, then $G_i=(A\cap G_i)(B\cap G_i)(A\cap G_i)$. Thus we may, and shall, assume that $A_i=(A\cap G_i)$ and $B_i=(B\cap G_i)$.

In this section and the next we shall study groups G of type III which satisfy the further condition that any proper subgroup or quotient group of G which is of type III is factorizable. We first obtain some information about factorizable ABA-groups.

PROPOSITION 6.1. *Let* $G=ABA$ *with* A *and* B *abelian and* $(|A|, |B|)=1$. *Suppose* $G\simeq PSL(2, 2^n)$, $n\geqq 2$. *Then*:

(i) $|A|=2^n+1$, $C_G(a)=A$ *for all* $a\in A^{\#}$, *and* $|B_0|=2$.

(ii) *The involution $b_0 \in B_0$ satisfies $b_0^{-1}ab_0 = a^{-1}$ for all $a \in A$. In particular, G is of type* III.

(iii) $|B| = 2^n$ *and* $C_G(b) = B$ *for all* $b \in B^{\#}$.

Proof. By 2.3, G is simple, $|G| = (2^n - 1)2^n(2^n + 1)$, and G possesses abelian Hall subgroups R_1, R_2, and R_3 of orders $2^n - 1$, 2^n, and $2^n + 1$ respectively such that the following conditions hold.

(a) $C_G(x) = R_i$ for all $x \in R_i^{\#}$, $i = 1, 2, 3$.

(b) $|N_G(R_i)| = 2|R_i|$, $i = 1, 2$.

(c) $R_i^g \cap R_i \neq 1$ implies $g \in N_G(R_i)$, $i = 1, 2, 3$.

By 4.1, $|B|$ is even as G is not solvable. Therefore $|A|$ is odd. Let $a \in A^{\#}$ be of prime power order. By replacing R_1 or R_2 by a suitable conjugate, we may assume $a \in R_j$, $j = 1$ or 2. Then $A \subseteq C_G(a) = R_j$ and, since R_j is abelian, $R_j \subseteq C_G(A) = AC_B(A) \subseteq AZ(G) = A$. Therefore $A = R_j$. Therefore $C_G(a) = A$ for all $a \in A^{\#}$ and $|N| = 2|A|$. Since $N = AB_0$, $|B_0| = 2$ and there exists an involution $b_0 \in B_0$ such that $C_A(b_0) = 1$. Since $|A|$ is odd, $A = C_A(b_0)\{a \in A \mid b_0^{-1}ab_0 = a^{-1}\}$. Therefore $b_0^{-1}ab_0 = a^{-1}$ for all $a \in A$.

By replacing R_3 by a conjugate, we may assume that $b_0 \in R_3$. Then $B \subseteq C_G(b_0) = R_3$. Let $aba_1 \in R_3$, $a, a_1 \in A$, $b \in B$. Then $aba_1 = b_0^{-1}(aba_1)b_0 = a^{-1}ba_1^{-1}$ and $b^{-1}a^2b = a_1^{-2}$. By (c), either $a^2 = a_1^2 = 1$, or $b \in B_0$. If $a^2 = a_1^2 = 1$, then $a = a_1 = 1$ and $aba_1 = b \in B$. If $b \in B_0$, then $b^{-1}ab = a_1^{-1}$ and $aba_1 = b \in B$. Hence $B = R_3$. Therefore $|B| = 2^n$ and $C_G(b) = B$ for all $b \in B^{\#}$.

If $A = R_1$, then $|A| = 2^n - 1$ and $(2^n - 1)2^n(2^n + 1) = |G| \leqq |A|^2|B| = (2^n - 1)^2 2^n$, a contradiction. Therefore $A = R_2$ and $|A| = 2^n + 1$.

LEMMA 6.2. *Let $G = ABA$ be a factorizable group of type* III. *Let $G = A_1B_1A_1 \times A_2B_2A_2 \times \cdots \times A_nB_nA_n$. Then*

(i) $A = A_1A_2 \cdots A_n$ *and* $B = B_1B_2 \cdots B_n$, *and*

(ii) *if $a \in A_i^{\#}$, $i \geqq 2$, then $C_B(a) = B_1B_2 \cdots B_{i-1}B_{i+1} \cdots B_n$.*

Proof. Let $G_i = A_iB_iA_i$, $1 \leqq i \leqq n$. Let $a \in A$. Then $a = x_1x_2 \cdots x_n$, $x_i \in G_i$, $1 \leqq i \leqq n$. Since A is abelian, $x_i \in C_{G_i}(A_i)$ and hence, by 6.1(i), $x_i \in A_i$, $2 \leqq i \leqq n$. It follows immediately that $x_1 \in A \cap G_1 = A_1$. Therefore $A = A_1A_2 \cdots A_n$. Similarly, using 6.1(iii), $B = B_1B_2 \cdots B_n$. Thus (i) holds. Statement (ii) is an immediate consequence of (i) and 6.1(i).

PROPOSITION 6.3. *Let $G = ABA$ be of type* III *and suppose H is a normal subgroup of G such that either $H \subseteq A$ or $H \subseteq B$. Suppose further that $\bar{G} = G/H = \bar{A}\bar{B}\bar{A}$ is factorizable. Then G is factorizable.*

Proof. Let $\bar{G} = \bar{A}_1\bar{B}_1\bar{A}_1 \times \bar{A}_2\bar{B}_2\bar{A}_2 \times \cdots \times \bar{A}_n\bar{B}_n\bar{A}_n$. If $\bar{G}$ is solvable, then G is solvable and therefore factorizable. We may therefore assume that $\bar{A}_i\bar{B}_i\bar{A}_i \neq 1$, $2 \leqq i \leqq n$. Let $\bar{G}_i = \bar{A}_i\bar{B}_i\bar{A}_i$, G_i be the inverse image in G of $\bar{G}_i$, A_i be the inverse image in A of $\bar{A}_i$, and B_i be the inverse image in B of $\bar{B}_i$, $1 \leqq i \leqq n$. Then $G_i = A_iB_iA_i \lhd G$, $1 \leqq i \leqq n$, and G_1 is solvable.

Let j be an integer, $2 \leqq j \leqq n$. Since $H \lhd G_j$, $C_{G_j}(H) \lhd G_j$. If $H \subseteq A$, then $H \subseteq A_j \subseteq C_{G_j}(H)$. If $H \subseteq B$, then $H \subseteq B_j \subseteq C_{G_j}(H)$. Thus, in either case, $C_{G_j}(H)/H \neq 1$. Since G_j/H is simple, $C_{G_j}(H) = G_j$. Thus G_j is a central extension of $PSL(2, 2^{m_j})$. By 6.1, $\bar{B}_j$ is a Sylow 2-subgroup of $\bar{G}_j$. Thus B_jH contains a Sylow 2-subgroup of G_j. A Sylow 2-subgroup of B_jH is contained in $B \cap B_jH$ and is therefore abelian. By 2.3(v), $G_j = H \times H_j$ with $H_j \simeq PSL(2, 2^{m_j})$. Since $G_j \lhd G$ and H is solvable, $H_j \lhd G$. Now if $H \subseteq A$, then $A_j = H(A_j \cap H_j)$ and, since $G_j/H_j \simeq H \subseteq A$, $B_j \subseteq H_j$. If $H \subseteq B$, then $B_j = H(B_j \cap H_j)$ and $A_j \subseteq H_j$. Therefore, in either case, $A_j = (A_j \cap H)(A_j \cap H_j)$ and $B_j = (B_j \cap H)(B_j \cap H_j)$. Let $x \in H_j$. Then $x = aba'$, $a, a' \in A_j$, $b \in B_j$. Let $a = a_1a_2$, $a' = a_3a_4$, $b = b_1b_2$, $a_1, a_3 \in A_j \cap H$, $a_2, a_4 \in A_j \cap H_j$, $b_1 \in B_j \cap H$, $b_2 \in B_j \cap H_j$. Then $x = a_1a_2b_1b_2a_3a_4 = (a_1b_1a_3)(a_2b_2a_4)$. Since $H \cap H_j = 1$, $a_1b_1a_3 = 1$ and $x = a_2b_2a_4$. Therefore $H_j = (A_j \cap H_j)(B_j \cap H_j)(A_j \cap H_j)$.

Let $H_1 = G_1 = A_1B_1A_1$. For $1 \leqq k, m \leqq n$, $k \neq m$, $[H_k, H_m] \subseteq H_k \cap H_m \subseteq H_k \cap H_m \cap H = 1$. For $g \in G$, $g = h_1h_2 \cdots h_n$ with $h_i \in H_i$, $1 \leqq i \leqq n$. Suppose also that $g = h_1'h_2' \cdots h_n'$ with $h_i' \in H_i$, $1 \leqq i \leqq n$. Then $h_i'h_i^{-1} \in H \cap H_i$, $1 \leqq i \leqq n$. Since $H \cap H_i = 1$, $2 \leqq i \leqq n$, $h_i = h_i'$, $2 \leqq i \leqq n$, and therefore $h_1 = h_i'$. Therefore $G = H_1 \times H_2 \times \cdots \times H_n$ whence G is factorizable.

In the remainder of this section we shall assume that $G = ABA$ is of type III and that any proper subgroup or factor group of G which is of type III is factorizable. We shall assume G contains a nontrivial normal subgroup H and under various hypotheses on H we shall show that G is factorizable.

Proposition 6.4. *If $H \subseteq A$ or $H \subseteq B$, then G is factorizable.*

Proof. Assume first that $H \subseteq A$. Let $K = H(A \cap Z(G))$. Then $K \lhd G$. Let $\bar{G} = G/K$. Then $\bar{G} = \bar{A}\bar{B}\bar{A}$. By 5.1(ii), $A = C_A(b_0) \times I_A(b_0)$. Therefore, since $C_A(b_0) \subseteq A \cap Z(G) \subseteq K$, $\bar{b}_0^{-1}\bar{a}\bar{b}_0 = \bar{a}^{-1}$ for all $\bar{a} \in \bar{A}$. Hence $\bar{G}$ is of type III. Therefore $\bar{G}$ is factorizable and by 6.3, G is factorizable.

Assume next that $H \subseteq B$. By what was done above, we may assume $A \cap Z(G) = 1$ so that $b_0^{-1}ab_0 = a^{-1}$ for all $a \in A$. Let $\tilde{G} = G/H$. Then $\tilde{G} = \tilde{A}\tilde{B}\tilde{A}$ and $\tilde{b}_0^{-1}\tilde{a}\tilde{b}_0 = \tilde{a}^{-1}$ for all $\tilde{a} \in \tilde{A}$. Therefore $\tilde{G}$ is of type III. Hence $\tilde{G}$ is factorizable and by 6.3, G is factorizable.

Proposition 6.5. *If H is of odd order, then G is factorizable.*

Proof. By 6.4, we may assume $A \cap Z(G) = B \cap Z(G) = 1$. In particular, $b_0^{-1}ab_0 = a^{-1}$ for all $a \in A$. Let $K = (AH)\langle b_0\rangle$. Then $K = A(B \cap K)A$ is a solvable group of type III, $|K|$ is twice an odd number, and $B \cap Z(K) \subseteq C_B(A) = 1$. By 5.2(i), $K = (A(B_0 \cap K))T$ where $T = (B \cap T)^A \lhd K$. It follows from 5.2(ii) that $C_T(A) = 1 = T \cap \langle b_0\rangle$. Hence $2|T| = |\langle b_0\rangle T|$ divides $|K|$. Therefore $|T|$ is odd. It follows from 5.4(ii) that $T = 1$. Thus $A \lhd K$. Therefore $AH = A(B_0 \cap H)$. Hence $H = (A \cap H)(B_0 \cap H)$. Therefore $A \cap H \lhd H$. Since $A \cap H$ is a Hall subgroup of H, $A \cap H$ char H. Hence $A \cap H \lhd G$. If $A \cap H \neq 1$, then by 6.4, G is factorizable. If $A \cap H = 1$, then $H \subseteq B$ and G is factorizable by 6.4.

PROPOSITION 6.6. *If $H=A^*B^*A^*$, $A^*\subseteq A$, $B^*\subseteq B$, is isomorphic to $PSL(2, 2^n)$, $n\geqq 2$, then G is factorizable.*

Proof. By 6.4, we may assume $A\cap Z(G)=B\cap Z(G)=1$. Hence $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$. We may also assume $G\neq H$.

Let $C=C_G(H)$. Then $C\lhd G$ and $CH=C\times H$. Let $G_1=AH$. Then $G_1=AB^*A$ and $C\cap G_1\lhd G_1$. Let $\bar{G}_1=G_1/(C\cap G_1)$. Then $\bar{G}_1=\bar{A}\bar{B}^*\bar{A}$, $\bar{H}\lhd\bar{G}_1$, $C_{\bar{G}_1}(\bar{H})=1$. By 2.3(vi), $\bar{G}_1$ is isomorphic to a subgroup of $P\Gamma L(2, 2^n)$ containing $PSL(2, 2^n)$. By 6.1, $|\bar{A}^*|=2^n+1$. By 2.3(iii), $C_{\bar{G}_1}(\bar{A}^*)=\bar{A}^*$. Therefore $\bar{A}=\bar{A}^*$. Hence $A=(A\cap C)A^*$. Similarly $B=(B\cap C)B^*$.

It follows immediately that $G=C\times H$ and $C=(A\cap C)(B\cap C)(A\cap C)$. Furthermore, $b_0=b_1b_2$, $b_1\in B\cap C$, $b_2\in B^*$. Now $b_1^2=1$ and $b_1^{-1}ab_1=a^{-1}$ for all $a\in A\cap C$. Since $G\neq H$ and $B\cap Z(G)=1$, $A\cap C\neq 1$. Therefore $b_1\neq 1$. Hence C is of type III. Since $C\subset G$, C is factorizable. Therefore $G=C\times H$ is also factorizable.

PROPOSITION 6.7. *If $H=A^*B^*A^*$, $A^*\subseteq A$, $B^*\subseteq B$, is solvable, then G is factorizable.*

Proof. It follows from 6.4 that we may assume G does not possess any nontrivial normal subgroups which are contained in A or in B. In particular, we may assume $A^*\neq 1$, $B^*\neq 1$, $A\cap Z(G)=1$ so that $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$, and $N_G(A^*)\subset G$. Let $N^*=N_G(A^*)$ and $B'=B\cap N^*$. Then $N^*=AB'A$. Since $b_0\in B'$, N^* is of type III. Therefore N^* is factorizable. Let $N^*=A_1B_1'A_1\times A_2B_2'A_2\times\cdots\times A_nB_n'A_n$.

By 3.2, $H=(A^*B_0^*)T$ where $B_0^*=B^*\cap N^*$ and $T=(B^*\cap T)^{A^*}$, and A^* is a Hall subgroup of H. For $g\in G$, $A^{*g}\subseteq H$. Since A^* is an abelian Hall subgroup of H, $A^{*g}=A^{*h}$ for a suitable element $h\in H$. Therefore $G=HN^*$. Thus if N^* is solvable, then G is solvable and therefore G is factorizable. We may therefore assume $A_2B_2A_2\neq 1$.

Let $a\in A^*$. By 6.2(i), $a=a_1a_2\cdots a_n$, $a_i\in A_i$, $1\leqq i\leqq n$. Since $a^g\in A$ for all $g\in N^*$, $A_2'=\langle a_2^g \mid g\in A_2B_2'A_2\rangle\subseteq A\cap A_2B_2'A_2=A_2$. Since $A_2B_2'A_2$ is simple, $A_2'=1$. Therefore $a_2=1$. Hence $A^*\subseteq A_1A_3\cdots A_n$. Therefore A^* and $T=(B^*\cap T)^{A^*}$ are both centralized by B_2'.

By 6.2(i), $b_0=b_1b_2\cdots b_n$, $b_i\in B_i$, $1\leqq i\leqq n$. Since $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$, $b_2^{-1}ab_2=a^{-1}$ for all $a\in A_2$. Let $t\in T$ and $a\in A_2^\#$. Then $t^a\in H$ and therefore $t^a=a't'b$, $a'\in A^*$, $t'\in T$, $b\in B_0^*$. Since A^* and T are centralized by B_2',

$$t^a = a't'b = a'^{b_2}t'^{b_2}b^{b_2} = (t^a)^{b_2} = (t^{b_2})^{a^{-1}} = t^{a^{-1}}.$$

Hence $t^{a^2}=t$. Since $|A|$ is odd, $t^a=t$. Therefore T is centralized by A_2. Hence $T\subseteq C_G(A_2B_2'A_2)$. Therefore $N_G(A_2B_2'A_2)\supseteq\langle AB'A, T\rangle=G$. By 6.6, G is factorizable.

PROPOSITION 6.8. *If $H=A^*B^*A^*$, $A^*\subseteq A$, $B^*\subseteq B$, is of type* III *and $H\subset G$, then G is factorizable.*

Proof. Since H is of type III and $H\subset G$, H is factorizable. Let

$$H = A_1^*B_1^*A_1^*\times A_2^*B_2^*A_2^*\times\cdots\times A_n^*B_n^*A_n^*.$$

If H is solvable, then G is factorizable by 6.7. We may therefore assume $A_2^* B_2^* A_2^* \neq 1$. Let $K = A_2^* B_2^* A_2^*$ and $a \in A$. Then $K^a \lhd H^a = H$. Since $K \cap K^a \supseteq A_2^*$, it is a nontrivial normal subgroup of both K^a and K. Since K is simple, $K = K^a$. Similarly $K = K^b$ for all $b \in B$. Therefore $K \lhd G$. By 6.6, G is factorizable.

Proposition 6.9. *Suppose* $H = (B \cap H)^A$ *and* $\bar{G} = G/H = \bar{B}_1 \times \bar{A}\bar{B}_2\bar{A}$ *with* $\langle \bar{B}_1, \bar{B}_2 \rangle \subseteq \bar{B}$ *and* $\bar{A}\bar{B}_2\bar{A} \cong PSL(2, 2^n)$, $n \geqq 2$. *Then* G *is factorizable.*

Proof. Assume $A \cap Z(G) = B \cap Z(G) = 1$. By 3.3, H is nilpotent. Hence $AH\langle b_0 \rangle$ is solvable. Let $K = AH\langle b_0 \rangle$. Then $K = A(B \cap K)A$, $B \cap Z(K) \subseteq C_B(A) \subseteq B \cap Z(G) = 1$, and $A \cap Z(K) \subseteq C_A(b_0) \subseteq A \cap Z(G) = 1$. If $A \lhd K$, then $[A, H] \subseteq A \cap H = 1$ and hence $H \subseteq C_B(A) = 1$. Since $H \neq 1$, $N_K(A) \subset K$. Let $m = |B \cap H|$. By 5.6, $|H| = m^2$. Therefore

$$|G| = |H|\,|\bar{B}_1|\,|\bar{A}\bar{B}_2\bar{A}| = m^2 |\bar{B}_1| (2^n - 1) 2^n (2^n + 1).$$

By 6.2(i), $\bar{B} = \bar{B}_1 \times \bar{B}_2$. By 6.1, $|\bar{B}_2| = 2^n$ and $|\bar{A}| = 2^n + 1$. Therefore $|A| = |\bar{A}| = 2^n + 1$ and $|B| = |B \cap H|\,|\bar{B}| = m |\bar{B}_1| 2^n$. Hence

$$m^2 |\bar{B}_1| (2^n - 1) 2^n (2^n + 1) = |G| \leqq |A|^2 |B| = (2^n + 1)^2 m |\bar{B}_1| 2^n.$$

Therefore $m(2^n - 1) \leqq (2^n + 1)$ and hence $2^n(m-1) \leqq m + 1$. Since $n \geqq 2$, it follows that $4(m-1) \leqq m + 1$. Thus $3m \leqq 5$. Hence $m = 1$ and therefore $H = 1$, a contradiction. Therefore either $A \cap Z(G) \neq 1$, or $B \cap Z(G) \neq 1$. By 6.4, G is factorizable.

Proposition 6.10. *If* b_0 *is the only involution in* B_0 *and* $H = (B \cap H)^A$, *then* G *is factorizable.*

Proof. By 6.4, we may assume $A \cap Z(G) = B \cap Z(G) = 1$. Therefore $b_0^{-1} a b_0 = a^{-1}$ for all $a \in A$ and hence $C_N(b_0) = B_0$. Since $[A, N \cap H] \subseteq A \cap H = 1$, $N \cap H \subseteq C_B(A) \subseteq B \cap Z(G) = 1$.

Let $\bar{G} = G/H$. Then $\bar{G} = \bar{A}\bar{B}\bar{A}$ and $\bar{b}_0^{-1} \bar{a} \bar{b}_0 = \bar{a}^{-1}$ for all $\bar{a} \in \bar{A}$. Therefore $\bar{G}$ is of type III and hence $\bar{G}$ is factorizable. Let $\bar{G} = \bar{A}_1\bar{B}_1\bar{A}_1 \times \bar{A}_2\bar{B}_2\bar{A}_2 \times \cdots \times \bar{A}_n\bar{B}_n\bar{A}_n$. If $\bar{G}$ is solvable, then G is solvable and therefore factorizable. We may therefore assume $\bar{A}_2\bar{B}_2\bar{A}_2 \neq 1$.

Let $\bar{b}$ be an involution in $N_{\bar{G}}(\bar{A})$, $b \in B$. Then $A^b \subseteq AH$. Since A and A^b are abelian Hall subgroups of AH, $A^b = A^x$, $x \in AH$. Therefore $b = gh$ with $g \in N\#$ and $h \in H$. Now $gh = b = b^{b_0} = g^{b_0} h^{b_0}$. Since $N \cap H = 1$, $g^{b_0} = g$. Since $C_N(b_0) = B_0$, $g \in B_0$. Now $g^2 h^g h = ghgh = b^2 \in H$. Therefore $g^2 \in H \cap B_0 = 1$. Since b_0 is the only involution in B_0, $g = b_0$. Therefore $\bar{b} = \bar{g} = \bar{b}_0$. Thus $\bar{b}_0$ is the only involution in $N_{\bar{B}}(\bar{A})$.

By 6.1(i), there exists an involution $\bar{b} \in N_{\bar{B}_2}(\bar{A}_2)$. By 6.2(i), $\bar{A} = \bar{A}_1\bar{A}_2 \cdots \bar{A}_n$. Therefore $\bar{b} \in N_{\bar{B}}(\bar{A})$ and hence $\bar{b} = \bar{b}_0$. Since $\bar{b}_0^{-1} \bar{a} \bar{b}_0 = \bar{a}^{-1}$ for all $\bar{a} \in \bar{A}$ and $\bar{b}_0 = \bar{b} \in \bar{B}_2 \subseteq C_{\bar{B}}(\bar{A}_1\bar{A}_3 \cdots \bar{A}_n)$, $\bar{A} = \bar{A}_2$. Therefore $\bar{G} = \bar{B}_1 \times \bar{A}\bar{B}_2\bar{A}$. By 6.9, G is factorizable.

7. A class of groups of type III.

In this section we shall prove the following result.

THEOREM 7.1. *Let $G=ABA$ be of type* III. *Suppose*

(1) *any proper subgroup or factor group of G which is of type* III *is factorizable, and*

(2) $C_G(a)\subseteq N$ *for all* $a\in A^{\#}$.

Then G is factorizable.

We begin with a lemma which will be used in this section and again in §8.

LEMMA 7.2. *Let $G=ABA$ be of type* III. *Suppose $C_G(a)$ is solvable for all $a\in A^{\#}$. Let $b_1\in B_0$ be an involution, $A_1=C_A(b_1)$, and $A_2=I_A(b_1)$. Then $A=A_1\times A_2$ and $C_G(b_1)=A_1BA_1$.*

Proof. By 5.1(ii), $A=A_1\times A_2$. Let $g=aba'\in C_G(b_1)$, $a, a'\in A$, $b\in B$. Let $a=a_1a_2$ and $a'=a_1'a_2'$, $a_i\in A_i$, $i=1, 2$. Then $a_1a_2ba_1'a_2'=g=b_1^{-1}gb_1=a_1a_2^{-1}ba_1'a_2'^{-1}$. Hence $b^{-1}a_2^2b=a_2'^{-2}$. Suppose $a_2'^{-2}=1$. Then $a_2^2=1$ and, since $|A|$ is odd, $a_2'=a_2=1$. Thus $g=a_1ba_1'$ in this case. Suppose next that $a_2'^{-2}\neq 1$. Let $C=C_G(a_2'^{-2})$. Then $C=A(B\cap C)A$ is solvable and $\langle A, A^b\rangle\subseteq C$. By 3.2, A is a Hall subgroup of C. Hence $A^b=A^c$, $c\in C$. Therefore $b\in CN$ and hence $g=aba'\in CN$. Now $CN=A(B\cap CN)A$ is solvable and $b_0\in CN$. By 5.2(vi), $g\in C_{CN}(b_1)=A_1(B\cap CN)A_1$ and hence $g=a_3b'a_3'$, $a_3, a_3'\in A_1$, $b'\in B$. Since $\langle A_1, B\rangle\subseteq C_G(b_1)$, $C_G(b_1)=A_1BA_1$.

We turn now to the proof of the theorem. It follows from 6.4 that we may assume G does not possess any nontrivial normal subgroups which are contained either in A or in B. In particular, $A\cap Z(G)=B\cap Z(G)=1$ and $N\subset G$. Since $A\cap Z(G)=1$, $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$. We break up the argument into a sequence of lemmas.

Let $n(X)$ be the number of involutions in the set X.

LEMMA 7.3. *There exist integers m, r, $s\geqq 1$ such that*

(i) $n(B)=n(B_0)+(n(B_0)+1)(2^m-1)$,

(ii) $|B/B_0|=2^mr\leqq(n(B_0)+1)(2^m-1)+1$,

(iii) $|A|-1=2^mrs\leqq 2(n(B_0)+1)(2^m-1)$, *and*

(iv) $|G|=|A|\,|B|(|A|-(|A|-1)/|B/B_0|)=(2^mrs+1)|B|(2^mrs+1-s)$.

Proof. Let $b_0, b_1, \ldots, b_t$ be a complete set of distinct coset representatives of B_0 in B. For $i\geqq 1$ and $a\in A^{\#}$, set $b_i^{-1}ab_i=a_1ba_2$, a_1, $a_2\in A$, $b\in B$. By 5.3(vi), $a_1=a_2\neq 1$ and $b\in B-B_0$ is an involution. In particular, $n(B-B_0)\geqq 1$. By rearranging the order of the b_i's, we may assume that $b_1, b_2, \ldots, b_k$ are involutions and that b_iB_0 contains no involutions if $i>k$.

Since B is abelian, $\{b_jB_0 \mid 0\leqq j\leqq k\}$ forms a subgroup of B/B_0 of order $k+1=2^m$ for a suitable integer $m\geqq 1$. Now $|B/B_0|=2^mr$ for a suitable integer $r\geqq 1$. Furthermore, $n(B-B_0)=(n(B_0)+1)k=(n(B_0)+1)(2^m-1)$. Therefore $n(B)=n(B_0)+(n(B_0)+1)(2^m-1)$. Thus (i) holds.

Suppose $b_i^{-1}ab_i=b_j^{-1}a'b_j$, a, $a'\in A^{\#}$. Then $b_jb_i^{-1}ab_ib_j^{-1}=a'$. By 5.3(ii), $b_ib_j^{-1}\in N$.

Hence $b_i=b_j$ and $a=a'$. Therefore

$$\begin{aligned}(|B/B_0|-1)|A^\#| &= |\{b_i^{-1}ab_i \mid 1 \leqq i \leqq n \text{ and } a\in A^\#\}| \\ &\leqq |\{a_1ba_1 \mid a_1\in A^\# \text{ and } b \text{ is an involution in } B-B_0\}| \\ &\leqq |A^\#|n(B-B_0) = |A^\#|(n(B_0)+1)(2^m-1).\end{aligned}$$

Therefore $|B/B_0|=2^m r\leqq(n(B_0)+1)(2^m-1)+1$. Thus (ii) holds.

By 5.3(v), $|G|=|A|\,|B|(|A|-(|A|-1)/|B/B_0|)$. In particular, $|B/B_0|$ divides $|A|-1$. Let $|A|-1=|B/B_0|s=2^m rs$. Then $|G|=(2^m rs+1)|B|(2^m rs+1-s)$. Thus (iv) holds.

Fix $i\geqq 1$. It follows from 5.3(vii) that the map $a\to b$ from $A^\#$ into the set of all involutions in $B-B_0$ determined by $b_i^{-1}ab_i=a_1ba_1$ is at most two-to-one. Therefore $|A|-1=2^m rs\leqq 2n(B-B_0)=2(n(B_0)+1)(2^m-1)$ and (iii) holds.

LEMMA 7.4. *Suppose $b\in B_0-\{b_0\}$ is an involution. Let $A_1=C_A(b)$ and let $C=C_G(b)$. Then*:

(i) $|C|=|A_1|\,|B|(|A|-(|A_1|-1)/|B/B_0|)$.

(ii) $|A_1|=k|B/B_0|+1$, $k=1$ *or* 2.

(iii) *Any involution in B_0 either centralizes A_1 or inverts the elements of A_1.*

Proof. By 7.2, $C=A_1BA_1$ and $A=A_1\times I_A(b)$. If $A_1=1$, then, since $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$, $bb_0\in C_B(A)\subseteq B\cap Z(G)=1$ and therefore $b=b_0$. Since this is not the case, $A_1\neq 1$. It follows from 5.3(ii) that $N_G(A_1)\subseteq N$. Since B is abelian, $B_0\subseteq N_G(A_1)$. Therefore $N_B(A_1)=B_0$ and $C_C(a)\subseteq N\cap C\subseteq N_C(A_1)$ for all $a\in A_1^\#$. Statement (i) now follows immediately from 5.3(v) applied to C.

Since $B\cap Z(G)=1$, $C\subset G$. Since $b_0\in C$, C is of type III and is therefore factorizable. Let $C=A_{11}B_1A_{11}\times A_{12}B_2A_{12}\times\cdots\times A_{1n}B_nA_{1n}$. Since $C_C(a)\subseteq N_C(A_1)$ for all $a\in A_1^\#$, either C is solvable, or $C=B_1\times A_1B_2A_1$. Suppose first that C is solvable. Then it follows from 5.3(viii) that $|A_1|=|B/B_0|+1$ and any involution in B_0 either centralizes A_1 or inverts each element of A_1. Suppose next that $C=B_1\times A_1B_2A_1$. Then $B_1\subseteq B_0$. Hence $B_0=B_1\times(B_2\cap B_0)$. By 6.1, $|A_1|=|B_2|+1$, $|B_2\cap B_0|=2$, and the involution in $B_2\cap B_0$ inverts A_1 elementwise. Therefore

$$|A_1| = |B_2|+1 = 2|B_2/(B_2\cap B_0)|+1 = 2|B_1B_2/B_1(B_2\cap B_0)|+1 = 2|B/B_0|+1,$$

and any involution in B_0 either centralizes A_1 or inverts each element of A_1. Thus (iii) and (iv) hold.

LEMMA 7.5. *b_0 is the only involution in B_0.*

Proof. Suppose there exists an involution $b_1\in B_0-\{b_0\}$. Let $A_1=C_A(b_1)$ and $A_2=I_A(b_1)$. Then $A=A_1\times A_2$. Furthermore, since $b_0^{-1}ab_0=a^{-1}$ for all $a\in A$, $A_2=C_A(b_1b_0)$. Let $b\in B_0$ be an involution. By 7.4(iii), either $C_A(b)=A$, $C_A(b)=A_1$, $C_A(b)=A_2$, or $C_A(b)=1$. Hence one of the elements b, bb_1, bb_1b_0, or bb_0 centralizes A. Since $C_B(A)\subseteq B\cap Z(G)=1$, we must have $b=1$, $bb_1=1$, $bb_1b_0=1$, or $bb_0=1$. Therefore b_0, b_1, and b_1b_0 are the only involutions in B_0. Therefore $n(B_0)=3$.

By 7.3(iii), $|A|-1=|B/B_0|s=2^m rs \leqq 8(2^m)-8$ with $m, r \geqq 1$. Now $A=A_1 \times A_2$ and by 7.4(ii), $|A_i|=k_i 2^m r+1$, $k_i=1$ or 2, $i=1, 2$. Therefore

$$(1) \qquad |A|-1 = (2^m r)(2^m r k_1 k_2 + k_1 + k_2) \leqq 8(2^m)-8.$$

Therefore $2^m r(2^m r+2) \leqq 8(2^m)-8$. Hence $8 \leqq 2^m(8-2^m r^2-2r)$. In particular, $8 > 2^m r^2+2r$. Since $m, r \geqq 1$, $2^m \leqq 4$ and $r=1$.

Suppose first that $2^m=2$. By (1), $2(2k_1k_2+k_1+k_2) \leqq 8$. Therefore $k_1=k_2=1$. Therefore $|B/B_0|=2^m r=2$, $|A_1|=2k_1+1=3$, and by (1), $|A|=9$. By 7.4(i), $|C_G(b_1)| =(3)|B|(2)$. On the other hand, by 7.3(iv), $|G|=(9)|B|(5)$. This contradicts the fact that $|C_G(b_1)|$ divides $|G|$.

Suppose next that $2^m=4$. By (1), $4(4k_1k_2+k_1+k_2) \leqq 24$. Therefore $k_1=k_2=1$. Therefore $|B/B_0|=4$, $|A_1|=5$, and $|A|=25$. Then $|C_G(b_1)|=(5)|B|(4)$ and $|G| =(25)|B|(19)$. Once again $|C_G(b_1)|$ does not divide $|G|$, a contradiction. Therefore b_0 is the only involution in B_0.

LEMMA 7.6. *The following conditions hold.*

(i) $|B/B_0|=2^m$.

(ii) $|A|=2^m+1$ *or* $2^{m+1}+1$.

(iii) *If* $|A|=2^{m+1}+1$, *then every involution in* G *is conjugate to* b_0.

Proof. By 7.5, $n(B_0)=1$. By 7.3(ii), $|B/B_0|=2^m r \leqq 2(2^m)-1$. Therefore $1 \leqq (2-r)2^m$ and hence $r=1$. Thus (i) holds. Furthermore, by 7.3(iii) and (iv), $|A|=2^m s+1$ and $|G|=(2^m s+1)|B|(2^m s+1-s)$. We shall determine the possible values for s by comparing the number of conjugates of b_0 with $n(G)$.

Let $a_1 b a_2$ be an involution, $a_1, a_2 \in A$, $b \in B$. Then $a_1 b a_2 = a_2^{-1} b^{-1} a_1^{-1}$. By 5.3(iv), either $a_1=a_2^{-1}$ and $b=b^{-1}$, or $b \in B_0$. Suppose $b \in B_0$. Then $b a_1^b a_2 = b^{-1}(a_2^{-1})^{b^{-1}} a_1^{-1}$ and therefore $b^2 \in A \cap B=1$. Since b_0 is the only involution in B_0, $b=b_0$. Therefore $a_1 b a_2 = a_1 a_2^{-1} b = a_3 b$, $a_3 \in A$. Since $|A|$ is odd, $a_3=a_4^2$, $a_4 \in A$. Therefore $a_1 b a_2 = a_3 b = a_4^2 b = a_4 b a_4^{-1}$. Thus every involution in G is of the form $a^{-1} b a$ with $a \in A$ and b an involution in B. Now suppose $a_1^{-1} b_1 a_1 = a_2^{-1} b_2 a_2$ where $a_1, a_2 \in A$, and b_1 and b_2 are involutions in B. Then $a_2 a_1^{-1} b_1 = b_2 a_2 a_1^{-1}$. By 5.3(iv), $b_1=b_2$ and either $a_1=a_2$, or $b_1 \in N$. In particular, $b_1 \in C_G(a_2 a_1^{-1})$. If $b_1 \in N$, then $b_1=b_0$. Since $C_A(b_0)=1$, this forces $a_2 a_1^{-1}=1$ so that $a_1=a_2$ in this case as well. Therefore $n(G)=|A|n(B)$. Hence, by 7.3(i),

$$n(G) = |A|(2^{m+1}-1).$$

By 7.2, $C_G(b_0)=B$. Thus the number of conjugates of b_0 in G is

$$|G|/|B| = (2^m s+1)(2^m s+1-s) = |A|(2^m s+1-s).$$

Since the number of conjugates of b_0 is at most $n(G)$, $(2^m s+1-s) \leqq (2^{m+1}-1)$. Thus $2-s \leqq 2^m(2-s)$ and therefore $s=1$ or 2. Therefore $|A|=2^m s+1=2^m+1$ or $2^{m+1}+1$ and (ii) holds. Furthermore, if $|A|=2^{m+1}+1$, then $s=2$ and hence $|A|(2^m s+1-s)=|A|(2^{m+1}-1)$. That is, if $|A|=2^{m+1}+1$, then the number of conjugates of b_0 is $n(G)$. Thus (iii) holds.

LEMMA 7.7. *If* $|A|=2^m+1$, *then* G *is factorizable.*

Proof. By 7.6(i), $|B/B_0|=2^m$. Therefore, by 7.3(iv), $|G|=|A|\,|B|2^m=|A|\,|B_0|2^{2m}$. Therefore B_0 contains Sylow p-subgroups of G for all odd primes p dividing $|B|$, and A is a Hall subgroup of G. Let $B^*=O(B_0)$.

Assume first that $B^*=1$. Then $|B_0|=2^k$ for some integer $k\geqq 1$, and $|G|=|A|2^{2m+k}$. Let S be a Sylow 2-subgroup of G. Then $G=AS$. Thus G is a product of two nilpotent groups and hence G is solvable [11]. Therefore G is factorizable.

Assume next that $B^*\neq 1$. Let $a_1b_1a_2\in N_G(B^*)$, $a_1, a_2\in A$, $b_1\in B$, and let $b\in B^*$. Then $a_2^{-1}b_1^{-1}a_1^{-1}ba_1b_1a_2=b'\in B^*$. Hence

$$a_2^{-1}b_1^{-1}a_1^{-1} = b'(a_2^{-1}b_1^{-1}a_1^{-1})b^{-1} = (a_2^{-1})^{b'^{-1}}b'b_1^{-1}b^{-1}(a_1^{-1})^{b^{-1}}.$$

Therefore

$$bb_1b'^{-1}(a_2^{b'^{-1}}a_2)b_1^{-1} = (a_1^{-1})^{b^{-1}}a_1.$$

It follows from 5.3(iii) that $b'^{-1}b=1$. Therefore $b=b'$. Therefore $B^*\subseteq Z(N_G(B^*))$.

B^* is an abelian Hall subgroup of G. Therefore τ, the transfer of G [illegible] maps $B^*=B^*\cap Z(N_G(B^*))$ onto itself [8]. Therefore $\ker(\tau)\subset G$. [illegible] $\subseteq\ker(\tau)$. Hence $\ker(\tau)$ is of type III. By 6.8, G is factorizable.

LEMMA 7.8. *If* $|A|=2^{m+1}+1$, *then* G *is factorizable.*

Proof. By 7.6(i), $|B/B_0|=2^m$. Therefore, by 7.3(iv), $|G|=(2^{m+1}+1)|$[illegible] Thus a Sylow 2-subgroup of G is contained in B and is therefore abelian. By 7.6(iii), every involution in G is conjugate to b_0. By 7.2, $C_G(b_0)=B$. Therefore the centralizer of any involution of G is abelian. If $O(G)\neq 1$, then G is factorizable by 6.5. We may therefore assume $O(G)=1$.

By 2.2, G is isomorphic to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$ where either $q\equiv 3$ or $5 \pmod 8$, $q\geqq 5$, or $q=2^n$, $n\geqq 2$. It follows from 2.3(iv) and 2.4(i) and (ii) that G possesses a normal subgroup $H\cong PSL(2, q)$ such that $|G/H|$ is odd. Since $|b_0|=2$, $b_0\in H$. Hence $a^2=b_0^{-1}(a^{-1}b_0a)\in H$ for all $a\in A$. Since $|A|$ is odd, $A\subseteq H$. Hence $H=A(B\cap H)A$. Since $b_0\in H$, H is of type III. If $H\subset G$, then G is factorizable by 6.8. We may therefore assume that $H=G$.

If $q\equiv 3$ or $5 \pmod 8$, $q>5$, then, by 2.4(iii), $B=C_G(b_0)$ is dihedral of order $q\pm 1$. Since $q\pm 1\neq 4$ in this case, B is not abelian, a contradiction. Therefore $G\cong PSL(2, q)$ where $q=5$ or $q=2^n$, $n\geqq 2$. By 2.4(ii), $PSL(2, 5)\cong PSL(2, 4)$. Thus $G\cong PSL(2, 2^n)$, $n\geqq 2$, in either case. Therefore G is factorizable.

It follows from 7.6(ii), 7.7, and 7.8 that G is factorizable. This completes the proof of Theorem 7.1.

8. The structure of groups of type III.

In this section we shall prove the following result.

THEOREM 8.1. *Every* ABA*-group of type* III *is factorizable.*

Proof. Assume false and let $G=ABA$ be a counterexample of minimal order. Then any proper subgroup or factor group of G which is of type III is factorizable.

It follows from 6.4 that $A \cap Z(G) = B \cap Z(G) = 1$. In particular, $b_0^{-1}ab_0 = a^{-1}$ for all $a \in A$.

For $a \in A^{\#}$, let $N^*(a) = \langle C_G(a), N\rangle$. Since $N = AB_0$ and $C_G(a) = AC_B(a)A$, $N^*(a) = A(C_B(a)B_0)A$. Let $L(a) = \langle a^b \mid b \in B_0\rangle$. Then $L(a) \lhd A$ and $B_0 \subseteq N_G(L(a))$. Since A and B are abelian, $\langle A, C_B(a)\rangle \subseteq C_G(L(a))$. Therefore $L(a) \lhd N^*(a)$. It follows from 6.4 that $N^*(a) \subset G$. Let $M(a)$ be a fixed, but arbitrary, maximal subgroup of G containing $N^*(a)$. Since $A \subseteq M(a)$, $M(a) = A(B \cap M(a))A$. Since $b_0 \in M(a)$, $M(a)$ is of type III. Hence $M(a)$ is factorizable. Furthermore, $A \cap Z(M(a)) \subseteq C_A(b_0) = 1$ and $B \cap Z(M(a)) \subseteq C_B(A) \subseteq B \cap Z(G) = 1$.

It follows from 7.1 that there exists an element $a^* \in A^{\#}$ such that $N^*(a^*) \nsubseteq N$. We shall reach a contradiction by studying $M(a^*)$. We break up the argument into a sequence of lemmas.

LEMMA 8.2. *Suppose $N^*(a) \nsubseteq N$, $a \in A^{\#}$. Then $M(a)$ is solvable.*

Proof. Let $M = M(a)$ and assume that M is not solvable. Let $A_iB_iA_i$, $1 \leqq i \leqq n$, be the components of M. Then $M = A_1B_1A_1 \times A_2B_2A_2 \times \cdots \times A_nB_nA_n$. Since $N^*(a) \nsubseteq N$ and M is not solvable, at least two of the A_i's are nontrivial. For otherwise $M = B_1 \times A_2B_2A_2$ and $N^*(a) \nsubseteq N$ by 6.1(i). By 6.2(i), $A = A_1A_2 \cdots A_n$ and $b_0 = b_1b_2 \cdots b_n$, $b_i \in B_i$, $1 \leqq i \leqq n$. Since $b_0^{-1}a'b_0 = a'^{-1}$ for all $a' \in A$, $b_i^{-1}a_ib_i = a_i^{-1}$ for all $a_i \in A_i$, $1 \leqq i \leqq n$. Hence $b_i \in N_{B_i}(A_i) \subseteq B_0$, $1 \leqq i \leqq n$, $C_A(b_2b_0) = A_2 \neq 1$, and $C_A(b_2) = A_1A_3 \cdots A_n \neq 1$.

We first show that $N^*(a') \subseteq M$ for all $a' \in A^{\#}$. Now if $N^*(a') \subseteq N$, then $N^*(a') \subseteq N^*(a) \subseteq M$. We may therefore assume $N^*(a') \nsubseteq N$. Let $M' = M(a')$.

Assume first that M' is solvable. Then $M' = (AV)B_0$ where $V = (B \cap V)^A \lhd M'$. Since $N^*(a') \nsubseteq N$, $N_{M'}(A) \subset M'$. By 5.7,

$$M' = ((A_1A_3 \cdots A_n)V_1 \times A_2V_2)B_0$$

where V_1 and V_2 are A-invariant subgroups of V and therefore satisfy $V_i = (B \cap V_i)^A$, $i = 1, 2$. Let $H_1 = A_2B_2A_2$. Let $H_2 = A_jB_jA_j$ be a nontrivial component of M with $\neq 2$. Then

$$N_G(H_1) \supseteq \langle M, C_B(A_2)\rangle \supseteq \langle M, B \cap V_1\rangle,$$

and

$$N_G(H_2) \supseteq \langle M, C_B(A_j)\rangle \supseteq \langle M, B \cap V_2\rangle.$$

Since G is not factorizable, it follows from 6.6 and 6.7 that $N_G(H_i) \subset G$, $i = 1, 2$. Since M is a maximal subgroup of G, $N_G(H_i) = M$, $i = 1, 2$. Therefore

$$\langle B \cap V_1, B \cap V_2\rangle \subseteq M.$$

Since $AB_0 \subseteq M$ and $V_i = (B \cap V_i)^A$, $i = 1, 2$, $M' \subseteq M$ and hence $N^*(a') \subseteq M$.

Assume next that M' is not solvable. Let $H_i' = A_i'B_i'A_i'$, $1 \leqq i \leqq m$, be the components of M'. Then $m \geqq 2$, $H_i' \neq 1, 2 \leqq i \leqq m$, and $M' = A_1'B_1'A_1' \times A_2'B_2'A_2' \times \cdots A_m'B_m'A_m'$. By 6.2(i), $A = A_1'A_2' \cdots A_m'$. Furthermore, since $b_2 \in B_0 \subseteq M'$, $b_2 = b_1'b_2' \cdots b_m'$ with $b_i' \in N_{B_i'}(A_i')$, $1 \leqq i \leqq m$.

Suppose $b_2=b_1'$. Then $A_2=C_A(b_2b_0)=C_A(b_1'b_0)\subseteq A_1'$ and $A_2'\cdots A_m'\subseteq C_A(b_1')=C_A(b_2)=A_1A_3\cdots A_n\subseteq C_A(B_2)$. In particular, $N_G(H_2')\supseteq\langle M', C_B(A_2')\rangle\supseteq\langle M', B_2\rangle$. By 6.6, $N_G(H_2')\subset G$. Since M' is a maximal subgroup of G, $N_G(H_2')=M'$. Therefore $B_2\subseteq M'$. Since $B_2\subseteq C_G(A_2'\cdots A_m')$, it follows from 6.2(ii) that $B_2\subseteq B_1'$. Therefore, since $A_2\subseteq A_1'$, $A_2B_2A_2\subseteq A_1'B_1'A_1'$. This is impossible as $A_1'B_1'A_1'$ is solvable and $A_2B_2A_2$ is not. Therefore, by rearranging the components of M', $b_2=b_1'b_2'\cdots b_j'$ with $b_i'\neq 1$, $2\leqq i\leqq j$. It follows from 6.1 that $b_i'^{-1}a_i'b_i'=a_i'^{-1}$ for all $a_i'\in A_i'$, $2\leqq i\leqq j$.

Suppose $j<m$. Then $A_2'\subseteq C_A(b_2b_0)=A_2\subseteq C_A(B_1B_3\cdots B_n)$, and $A_{j+1}'\subseteq C_A(b_2)=A_1A_3\cdots A_n\subseteq C_A(B_2)$. Therefore $N_G(H_2')\supseteq\langle M', B_1B_3\cdots B_n\rangle$ and $N_G(H_{j+1}')\supseteq\langle M', B_2\rangle$. By 6.6, $N_G(H_2')\subset G$ and $N_G(H_{j+1}')\subset G$. Therefore $N_G(H_2')=N_G(H_{j+1}')=M'$. Hence $B_1B_2\cdots B_n\subseteq M'$. Since $A\subseteq M'$, $M\subseteq M'$. Since M is a maximal subgroup of G, $M=M'$. In particular, $N^*(a')\subseteq M$.

Suppose $j=m$. Then $A_1A_3\cdots A_n=C_A(b_2)\subseteq A_1'$, and $A_2'A_3'\cdots A_m'\subseteq C_A(b_2b_0)=A_2\subseteq C_A(B_1B_3\cdots B_n)$. In particular, $N_G(H_2')\supseteq\langle M', B_1B_3\cdots B_n\rangle$. Therefore $B_1B_3\cdots B_n\subseteq M'$. Since $B_1B_3\cdots B_n$ centralizes $A_2'A_3'\cdots A_m'$, it follows from 6.2(ii) that $B_1B_3\cdots B_n\subseteq B_1'$. Since $A_1A_3\cdots A_n\subseteq A_1'$ and $A_1'B_1'A_1'$ is solvable, we must have $n=2$. Since at least two of the A_i's are nontrivial, $A_1B_1A_1\neq 1$. By expressing b_2' as a product of elements in the B_i's and repeating the above argument, we see that either $M=M'$ or $m=2$, $B_1'\subseteq B_1$, and $A_1'\subseteq A_1$. In the latter case, $A_1B_1A_1=A_1'B_1'A_1'$ and hence $N_G(A_1B_1A_1)\supseteq\langle M, M'\rangle$. It follows from 6.7 that $N_G(A_1B_1A_1)\subset G$. Therefore $M=N_G(A_1B_1A_1)=M'$. Thus $M=M'$ in either case. Hence $N^*(a')\subseteq M$.

Therefore $N^*(a')\subseteq M$ for all $a'\in A^{\#}$. Now suppose $A\cap A^b\neq 1$, $b\in B$. Then $A^b\subseteq C_G(A\cap A^b)\subseteq M$. Therefore $M^b=A^b(B\cap M)^bA^b\subseteq M$ and hence $b\in N_G(M)$. It follows from 6.8 that $N_G(M)\subset G$. Therefore $N_G(M)=M$ and hence $b\in M$.

Let $b^*=b_1b_3\cdots b_n$. Then $b^*=b_2b_0$. Let $C=C_G(b^*)$. We next show that $C=A_2BA_2$. Clearly $\langle A_2, B\rangle\subseteq C$. Thus it suffices to show that each element of C can be expressed in the form a_2ba_2' with $a_2, a_2'\in A_2$ and $b\in B$. Let $a''ba'\in C$, $a'', a'\in A$, $b\in B$. Let $a''=a_1a_2\cdots a_n$ and $a'=a_1'a_2'\cdots a_n'$ with $a_i, a_i'\in A_i$, $1\leqq i\leqq n$. Then

$$\begin{aligned} a_1a_2\cdots a_nba_1'a_2'\cdots a_n' &= (a_1a_2\cdots a_nba_1'a_2'\cdots a_n')^{b^*} \\ &= a_1^{-1}a_2a_3^{-1}\cdots a_n^{-1}ba_1'^{-1}a_2a_3'^{-1}\cdots a_n'^{-1}. \end{aligned}$$

Therefore $b^{-1}a_1^2a_3^2\cdots a_n^2b=a_1'^{-2}a_3'^{-2}\cdots a_n'^{-2}$. Thus either $a_1a_3\cdots a_n=a_1a_3'\cdots a_n'=1$, or $A\cap A^b\neq 1$. In the first case, $a''ba'=a_2ba_2'$ is of the required form. Suppose, on the other hand, that $A\cap A^b\neq 1$. Then $b\in M$. Let $b=g_1g_2\cdots g_n$, $g_i\in B_i$, $1\leqq i\leqq n$. Then $g_i^{-1}a_i^2g_i=a_i'^{-2}$, $i=1, 3, \dots, n$. By 5.2(v), $g_1^{-1}a_1g_1=a_1'^{-1}$. By 6.1(i), either $a_i'=1$, or $g_i^{-1}a_ig_i\in C_{A_iB_iA_i}(a_i'^{-2})=A_i$, $3\leqq i\leqq n$. If $a_i'=1$, then $a_i=1$ and therefore $g_i^{-1}a_ig_i=1=a_i'^{-1}$. If $g_i^{-1}a_ig_i\in A_i$, then, since $|A_i|$ is odd, $g_i^{-1}a_ig_i=a_i'^{-1}$ being the unique square root in A_i of $a_i'^{-2}$. Therefore $b^{-1}a_ib=g_i^{-1}a_ig_i=a_i'^{-1}$, $i=1, 3, \dots, n$. Hence

$$a''ba' = a_2bb^{-1}(a_1a_3\cdots a_n)ba_1'a_3'\cdots a_n'a_2' = a_2ba_2'.$$

Therefore $C=A_2BA_2$.

Since $B\cap Z(G)=1$, $C\subset G$. Since $b_0\in C$, C is of type III. Therefore C is factorizable. Let $C=A_{21}B_{21}A_{21}\times A_{22}B_{22}A_{22}\times\cdots\times A_{2k}B_{2k}A_{2k}$. Since $A_2B_2A_2\subseteq C$, C is not solvable. Hence $A_{22}B_{22}A_{22} \neq 1$. Suppose $A_{2j} \neq 1$ for some $j \neq 2$, $1 \leqq j \leqq k$. Then $B_{22} \subseteq C_G(A_{2j}) \subseteq M$. Since B_{22} centralizes $A_{2j} \subseteq A_2$, it follows from 6.2(ii) that $B_{22} \subseteq B_1 B_3 \cdots B_n \subseteq C_G(A_2)$. This is impossible as $A_{22} \subseteq A_2$, $A_{22} \neq 1$, and $C_{A_{22}}(B_{22}) = 1$. Therefore $A_2 = A_{22}$ and $C = B_{21} \times A_2 B_{22} A_2$. Since $C/A_2 B_{22} A_2$ is solvable, $A_2 B_2 A_2 \subseteq A_2 B_{22} A_2$. It follows from 6.1 that $A_2 B_2 A_2 = A_2 B_{22} A_2$. But $B_{21} \subseteq C_G(A_2) \subseteq M$. Therefore $B \subseteq C = B_{21} \times A_2 B_2 A_2 \subseteq M$. Since $A \subseteq M$, $M = G$, a contradiction. This contradiction completes the proof of Lemma 8.2.

LEMMA 8.3. *b_0 is the only involution in B_0 and $O(B_0) = 1$.*

Proof. Assume false. Recall that there exists an element $a^* \in A^\#$ such that $N^*(a^*) \nsubseteq N$. Let $M^* = M(a^*)$. By 8.2, M^* is solvable. Therefore $M^* = (AT)B_0$ where $T = (B \cap T)^A \lhd M^*$. Since $N^*(a^*) \nsubseteq N$, $N_{M^*}(A) \subset M^*$ and $T \neq 1$. Furthermore, $A \cap Z(M^*) \subseteq C_A(b_0) = 1$ and $B \cap Z(M^*) \subseteq C_B(A) = 1$. Suppose there exists an involution $b' \in B_0 - \{b_0\}$. Then, by 5.7, $M^* = (A_1 T_1 \times A_2 T_2) B_0$ where $A_1 = C_A(b')$, $A_2 = C_A(b' b_0)$, and T_1 and T_2 are A-invariant subgroups of T. Furthermore, $A = A_1 \times A_2$ and, since $B \cap Z(G) = 1$, A_1 and A_2 are both nontrivial. We may assume, without loss of generality, that $T_1 \neq 1$. In this case we set $B_1 = \langle b' \rangle$. Suppose next that b_0 is the only involution in B_0. Then $O(B_0) \neq 1$. By 5.9, $M^* = (A_1 T \times A_2) B_0$ where $A_1 = C_A(O(B_0))$ and $A_2 = [A, O(B_0)]$. Here again, $A = A_1 \times A_2$ and A_1 and A_2 are both nontrivial. In this case we set $B_1 = O(B_0)$, $T_1 = T$, and $T_2 = 1$. Thus, in either case, $M^* = (A_1 T_1 \times A_2 T_2) B_0$ where $A_1 = C_A(B_1)$, $A = A_1 \times A_2$, A_1, A_2, and T_1 are nontrivial, and T_1 and T_2 are A-invariant subgroups of T.

Since $T = (B \cap T)^A$, $C_T(A) \subseteq C_{B \cap T}(A) \subseteq B \cap Z(G) = 1$. Hence, by 5.4(v), T is an elementary abelian 2-group. Let $T_1 = T_{11} \times T_{12} \times \cdots \times T_{1m}$ where T_{1i} is A-irreducible, $1 \leqq i \leqq m$. Now if V is any A-invariant subgroup of T, then $V = (B \cap V)^A$ and therefore $V \lhd M^*$. Hence $T_i = (B \cap T_i)^A \lhd M^*$, $i = 1, 2$, and $T_{1j} = (B \cap T_{1j})^A \lhd M^*$, $1 \leqq j \leqq m$. It follows that

$$C_T(b_0) = C_{T_1}(b_0) \times C_{T_2}(b_0) = C_{T_{11}}(b_0) \times \cdots \times C_{T_{1m}}(b_0) \times C_{T_2}(b_0).$$

By 5.4(i), $C_T(b_0) = B \cap T$. Therefore

$$B \cap T = (B \cap T_1) \times (B \cap T_2) = (B \cap T_{11}) \times \cdots \times (B \cap T_{1m}) \times (B \cap T_2).$$

Since $B \cap Z(M^*) = 1$, it follows from 5.2(ii) and (iii) that $B \cap M^* = (B \cap T) \times B_0$. Hence $B \cap M^* = (B \cap T_1) \times (B \cap T_2) \times B_0$.

We next argue that $N^*(a) \subseteq M^*$ for all $a \in A^\#$. If $N^*(a) \subseteq N$, then $N^*(a) \subseteq M^*$. Suppose $N^*(a) \nsubseteq N$, $a \in A^\#$. Then $M(a)$ is solvable by 8.2. Therefore $M(a) = (AW)B_0$ where $W = (B \cap W)^A \lhd M(a)$. As above, $M(a) = (A_1 W_1 \times A_2 W_2) B_0$ with $W_i = (B \cap W_i)^A$, $i = 1, 2$. Now $A_i T_i = A_i (B \cap T_i) A_i$, $i = 1, 2$. Therefore $N_G(A_1 T_1) \supseteq \langle M^*, C_B(A_1) \rangle \supseteq \langle M^*, B \cap W_2 \rangle$, and $N_G(A_2 T_2) \supseteq \langle M^*, C_B(A_2) \rangle \supseteq \langle M^*, B \cap W_1 \rangle$. It follows from 6.7 that $N_G(A_i T_i) \subset G$, $i = 1, 2$. Therefore $N_G(A_i T_i) = M^*$, $i = 1, 2$. Therefore $\langle B \cap W_1, B \cap W_2 \rangle \subseteq M^*$. Since $AB_0 \subseteq M^*$ and $W_i = (B \cap W_i)^A$, $i = 1, 2$, $M(a) \subseteq M^*$. Hence $N^*(a) \subseteq M^*$.

Suppose $A \cap A^b \neq 1$, $b \in B$. Then $A^b \subseteq C_G(A \cap A^b) \subseteq M^*$. Since A is an abelian Hall subgroup of M^*, $A^b = A^x$ for some $x \in M^*$. Therefore $b \in M^*N = M^*$. Suppose next that $A^b \cap M^* \neq 1$. Then $A^b \cap M^* \subseteq A^x$ for some $x \in M^*$. Therefore $A^b \subseteq C_G(A^b \cap M^*) \subseteq M^{*x} = M^*$. As above, $b \in M^*N = M^*$.

Let $C = C_G(B_1)$. We next show that $C = A_1BA_1$. It follows from 8.2 that $C_G(a)$ is solvable for all $a \in A^\#$. Thus, if $B_1 = \langle b' \rangle$, b' an involution, then $C = A_1BA_1$ by 7.2. Suppose $B_1 = O(B_0)$. Let $g = aba' \in C$, $a, a' \in A$, $b \in B$, and let $b^* \in B_1$. Let $a = a_1a_2$ and $a' = a_1'a_2'$ with $a_i, a_i' \in A_i$, $i = 1, 2$. Then

$$a_1a_2ba_1'a_2' = g = g^{b^*} = (a_1a_2ba_1'a_2')^{b^*} = a_1a_2^{b^*}ba_1'a_2'^{b^*}.$$

Hence $b^{-1}a_2(a_2^{-1})^{b^*}b = a_2'^{b^*}a_2'^{-1}$. Suppose first that $a_2'^{b^*}a_2'^{-1} = 1$ for all $b^* \in B_1$. Then $a_2(a_2^{-1})^{b^*} = 1$ for all $b^* \in B_1$. Hence $a_2, a_2' \in A_2 \cap A_1 = 1$ and therefore $g = a_1ba_1'$. Suppose next that $a_2'^{b^*}a_2'^{-1} \neq 1$ for some $b^* \in B_1$. Then $A^b \cap A \neq 1$ and therefore $b \in M^*$. Hence $g \in C_{M^*}(B_1)$. By 5.9(iii), $g \in A_1(B \cap M^*)A_1$. Therefore $C = A_1BA_1$.

Let $B_2 = N_B(A_1)$ and $B_3 = C_B(A_1)$. If $b \in B_2$, then $A^b \cap A \neq 1$ and therefore $b \in M^*$. Since $B \cap M^* = (B \cap T_1) \times (B \cap T_2) \times B_0$ and $(B \cap T_2)B_0 \subseteq B_2$, $B_2 = (B_2 \cap T_1)(B \cap T_2)B_0$. Since $[B_2 \cap T_1, A_1] \subseteq T_1 \cap A_1 = 1$,

$$B_2 \cap T_1 \subseteq C_{B \cap T_1}(A_1) \subseteq C_B(A) = 1.$$

Therefore $B_2 = (B \cap T_2)B_0$ and hence $B_3 = (B \cap T_2)(B_3 \cap B_0)$.

Since $B \cap Z(G) = 1$ and $B_1 \neq 1$, $C \subset G$. Since $b_0 \in C$, C is of type III. Therefore C is factorizable. Let $C = A_{11}B_{11}A_{11} \times A_{12}B_{12}A_{12} \times \cdots \times A_{1k}B_{1k}A_{1k}$. By 8.2, $C_G(a)$ is solvable for all $a \in A^\#$. Therefore either $C = B_{11} \times A_1B_{12}A_1$, or C is solvable. Suppose $C = B_{11} \times A_1B_{12}A_1$. It follows from 6.1(i) that $|A_1| = |B_{12}| + 1$ and $B_{11} = C_B(A_1) = B_3$. Furthermore, if $b \in C_{B \cap T_1}(a)$, $a \in A_1^\#$, then $b \in B_{11} \cap T_1 \subseteq C_{B \cap T_1}(A_1) \subseteq C_B(A) = 1$. Therefore A_1 acts regularly on T_1. By 5.2(vii), $C_{(A_1T_1)B_0}(a) \subseteq A_1B_0$ for all $a \in A_1^\#$. By 5.3(viii),

$$|A_1| = |(B \cap T_1)B_0/B_0| + 1 = |B \cap T_1| + 1.$$

Therefore $|B_{12}| = |B \cap T_1|$. Hence

$$|B| = |B_{11}|\,|B_{12}| = |B_3|\,|B \cap T_1| \leq |B \cap T_2|\,|B_0|\,|B \cap T_1| = |B \cap M^*|.$$

Therefore $B \subseteq M^*$. Since $A \subseteq M^*$, $M^* = G$, a contradiction. Therefore C is solvable.

Since C is solvable, $C = (A_1U)B_2$ where $U = (B \cap U)^{A_1} \lhd C$. Let $\bar{C} = C/B_3$. Then

$$\bar{C} = \bar{A}_1\bar{B}\bar{A}_1 = \bar{A}_1\bar{U}\bar{B}_2 \quad \text{and} \quad \bar{U} = (\overline{B \cap U})^{\bar{A}_1} = (\bar{B} \cap \bar{U})^{\bar{A}_1}.$$

Suppose $\bar{b}^{-1}\bar{a}\bar{b} = \bar{a}'$, $a, a' \in A_1$, $b \in B$. Then $b^{-1}ab = a'b^*$, $b^* \in B_3$. Hence $|b^{-1}ab| = |a'|\,|b^*|$ and therefore $b^* = 1$. In particular, $\bar{B} \cap Z(\bar{C}) \subseteq C_{\bar{B}}(\bar{A}_1) = 1$, $N_{\bar{B}}(\bar{A}_1) = \bar{B}_2 \subset \bar{C}$, and $\bar{A}_1 \cap Z(\bar{C}) \subseteq C_{\bar{A}_1}(\bar{b}_0) = 1$.

Since $\bar{U} = (\bar{B} \cap \bar{U})^{\bar{A}_1}$, $C_{\bar{U}}(\bar{A}_1) \subseteq C_{\bar{B}}(\bar{A}_1) \subseteq \bar{B} \cap Z(\bar{C}) = 1$. Hence by 5.4(v), $\bar{U}$ is an elementary abelian 2-group. Let $\bar{U} = \bar{U}_1 \times \bar{U}_2 \times \cdots \times \bar{U}_n$ where each $\bar{U}_j$ is $\bar{A}_1$-irreducible, $1 \leqq j \leqq n$. Then $\bar{U}_j = (\bar{B} \cap \bar{U}_j)^{\bar{A}_1} \lhd \bar{C}$, $1 \leqq j \leqq n$, and $\bar{B} \cap \bar{U} = (\bar{B} \cap \bar{U}_1)$

$\times(\bar{B}\cap\bar{U}_2)\times\cdots\times(\bar{B}\cap\bar{U}_n)$. Let U_j be the inverse image in U of $\bar{U}_j$, $1\leqq j\leqq n$. Then $U=U_1U_2\cdots U_n$, and $U_j=(B\cap U_j)^{A_1}$, $1\leqq j\leqq n$. If $U\subseteq M^*$, then $B\subseteq C\subseteq M^*$ and therefore $M^*=G$. Since this is not the case, $U\nsubseteq M^*$. Thus we may assume, by rearranging the U_j's that $U_1\nsubseteq M^*$.

Suppose $C_{\bar{A}_1}(\bar{U}_i)\neq 1$ for some i, $1\leqq i\leqq n$. Let $b\in B\cap U_i$ and $\bar{a}\in C_{\bar{A}_1}(\bar{U}_i)$, $a\in A_1^{\#}$. Then $\bar{b}^{-1}\bar{a}\bar{b}=\bar{a}$ and therefore $b^{-1}ab=a$. Hence $b\in C_G(a)\subseteq M^*$. Therefore $B\cap U_i\subseteq M^*$. Since $A_1\subseteq M^*$ and $U_i=(B\cap U_i)^{A_1}$, $U_i\subseteq M^*$. Hence, since $U_1\nsubseteq M^*$, $C_{\bar{A}_1}(\bar{U}_1)=1$. It follows from 5.4(iii) that $\bar{A}_1$ acts regularly on $\bar{U}_1$. It follows from 5.4(iii) and (iv) that $C_{\bar{A}_1}(\bar{U}_j)\neq 1$, $2\leqq j\leqq n$. Therefore $U_2U_3\cdots U_n\subseteq M^*$. Let $\bar{B}_4$ be a Sylow 2-subgroup of $\bar{B}_2$ and $\bar{B}_5=\Omega_1(\bar{B}_4)$. Since $C_{\bar{A}_1}(\bar{U}_1)=1$ and $C_{\bar{B}}(\bar{A}_1)=1$, $|\bar{B}_5|=2$ by 5.5(ii). Hence $\bar{b}_0$ is the only involution in $\bar{B}_2$.

Let $A_{11}=C_{A_1}(T_{11})$. Suppose $A_{11}=1$. Then, by 5.4(iii), A_1 acts regularly on T_{11}. Let $\bar{b}\in\bar{B}\cap\bar{U}_i$, $b\in B$, $i\geqq 2$. Then there exists an element $a\in A_1^{\#}$ such that $\bar{b}=\bar{b}^{\bar{a}}$. Hence $b=b^a$. Since $b\in U_i\subseteq M$, $b=b_{11}\cdots b_{1m}b_2b^*$ with $b^*\in B_0$, $b_2\in B\cap T_2$, and $b_{1j}\in B\cap T_{1j}$, $1\leqq j\leqq m$. Since $b=b^a$ and T is abelian,

$$b^*(b^{*a})^{-1} = b_{11}^a b_{11}^{-1}\cdots b_{1m}^a b_{1m}^{-1} b_2^a b_2^{-1}\in B_0\cap T\subseteq C_B(A) = 1.$$

Therefore $b_{11}^a=b_{11}$. Since A_1 acts regularly on T_{11}, $b_{11}=1$. Hence $b=b_{12}\cdots b_{1m}b_2b^*$ Since $B_2=(B\cap T_2)B_0$, it follows that

$$(\bar{B}\cap\bar{U}_2)(\bar{B}\cap\bar{U}_3)\cdots(\bar{B}\cap\bar{U}_n)\bar{B}_2\subseteq(\overline{B\cap T_{12}})\cdots(\overline{B\cap T_{1m}})(\overline{B\cap T_2})\bar{B}_0.$$

Hence

$$|\bar{B}|\leqq|\bar{B}\cap\bar{U}_1|\,|(\overline{B\cap T_{12}})\cdots(\overline{B\cap T_{1m}})(\overline{B\cap T_2})\bar{B}_0|.$$

Since A_1 and $\bar{A}_1$ act regularly on T_{11} and $\bar{U}_1$ respectively, it follows from 5.2(vii) and 5.3(viii) that $|A_1|=|B\cap T_{11}|+1$ and $|\bar{A}_1|=|\bar{B}\cap\bar{U}_1|+1$. Since $A\cap B_3=1=T_{11}\cap B_3$,

$$|A_1| = |\bar{A}_1| \quad\text{and}\quad |B\cap T_{11}| = |\overline{B\cap T_{11}}|.$$

Therefore

$$|\bar{B}\cap\bar{U}_1| = |\overline{B\cap T_{11}}|.$$

Hence

$$|\bar{B}|\leqq|\overline{B\cap T_{11}}|\,|(\overline{B\cap T_{12}})\cdots(\overline{B\cap T_{1m}})(\overline{B\cap T_2})\bar{B}_0| = |\overline{B\cap M^*}|.$$

Therefore

$$\bar{B} = \overline{B\cap M^*}.$$

Since $B_3\subseteq B\cap M^*$, $B=B\cap M^*$. Since $A\subseteq M^*$, $M^*=G$, a contradiction. Therefore $A_{11}\neq 1$.

Let $b\in B\cap T_{11}$ and set $\bar{b}=\bar{b}_1\bar{b}_2\cdots\bar{b}_n\bar{b}^*$ with $\bar{b}_i\in\bar{B}\cap\bar{U}_i$, $1\leqq i\leqq n$, and $\bar{b}^*\in\bar{B}_2$. Let $a\in A_{11}^{\#}$. Then $\bar{b}=\bar{b}^{\bar{a}}$ and therefore

$$\bar{b}^*(\bar{b}^{*\bar{a}})^{-1} = \bar{b}_1^{\bar{a}}\bar{b}_1^{-1}\cdots\bar{b}_n^{\bar{a}}\,\bar{b}_n^{-1}\in\bar{B}_2\cap\bar{U}\subseteq C_{\bar{B}}(\bar{A}_1) = 1.$$

Hence $\bar{b}^* = \bar{b}^{*\bar{a}}$ and $\bar{b}_i = \bar{b}_i^{\bar{a}}$, $1 \leqq i \leqq n$. Since $\bar{A}_1$ acts regularly on $\bar{U}_1$, $\bar{b}_1 = 1$. Since T is an elementary abelian 2-group, $b^2 = 1$ and hence $\bar{b}^{*2} = 1$. Since $\bar{b}_0$ is the only involution in $\bar{B}_2$, $C_{\bar{A}_1}(\bar{b}_0) = 1$, and $\bar{a} \in C_{\bar{A}_1}(\bar{b}^*)$, we must have $\bar{b}^* = 1$. Thus $\bar{b} = \bar{b}_2 \cdots \bar{b}_n$. Hence

$$\overline{B \cap T_{11}} \subseteq (\bar{B} \cap \bar{U}_2) \cdots (\bar{B} \cap \bar{U}_n).$$

Since $T_{11} = (B \cap T_{11})^A$, $A = A_1 \times A_2$, and A_2 centralizes T_{11}, we have $T_{11} = (B \cap T_{11})^{A_1}$. Therefore $\bar{T}_{11} \subseteq (\bar{B} \cap \bar{U}_2)^{\bar{A}_1} \cdots (\bar{B} \cap \bar{U}_n)^{\bar{A}_1} = \bar{U}_2 \cdots \bar{U}_n$. Hence $[T_{11}, U_1] \subseteq B_3$.

Let $t \in T_{11}$, $u \in U_1$, and $a \in A_{11}^{\#}$. Let $g = u^{-1}t^{-1}ut$. Then $g \in [T_{11}, U_1] \subseteq B_3 = C_B(A_1) \subseteq C_B(a)$. Therefore

$$u^{-1}t^{-1}ut = g = g^a = (u^{-1})^a(t^{-1})^a u^a t^a = (u^{-1})^a t^{-1} u^a t.$$

Hence $u^a u^{-1} \in C_{U_1}(t)$. Since t was arbitrary, $u^a u^{-1} \in C_{U_1}(T_{11})$. Let $V = C_{U_1}(T_{11})$. Then V is A_1-invariant. Hence either $\bar{V} = 1$, or $\bar{V} = \bar{U}_1$. Suppose $\bar{V} = 1$. Then $\bar{u}^{\bar{a}}\bar{u}^{-1} = 1$. Since $\bar{A}_1$ acts regularly on $\bar{U}_1$, $\bar{u} = 1$. Since u was chosen arbitrarily, this need not be the case. Therefore $\bar{V} = \bar{U}_1$. Hence $U_1 = VB_3$. Since $T_{11} = (B \cap T_{11})^{A_1}$, $B_3 \subseteq C_G(T_{11})$. Therefore $U_1 \subseteq C_G(T_{11})$. Hence $N_G(T_{11}) \supseteq \langle M^*, U_1 \rangle$. Since $U_1 \nsubseteq M^*$ and M^* is a maximal subgroup of G, $N_G(T_{11}) = G$.

Let $\tilde{G} = G/T_{11}$. Then $G = \tilde{A}\tilde{B}\tilde{A}$ and $\tilde{b}_0^{-1}\tilde{a}\tilde{b}_0 = \tilde{a}^{-1}$ for all $\tilde{a} \in \tilde{A}$. Hence $\tilde{G}$ is of type III. Therefore $\tilde{G}$ is factorizable. Let $\tilde{G} = \tilde{A}_1'\tilde{B}_1'\tilde{A}_1' \times \tilde{A}_2'\tilde{B}_2'\tilde{A}_2' \times \cdots \times \tilde{A}_r'\tilde{B}_r'\tilde{A}_r'$. Since G is not solvable, $\tilde{G}$ is not solvable. Therefore $\tilde{A}_2'\tilde{B}_2'\tilde{A}_2' \neq 1$. Suppose $\tilde{A}_s' \neq 1$, $s \neq 2$. Let A_s' be the inverse image in A of $\tilde{A}_s'$ and let B_2' be the inverse image in B of $\tilde{B}_2'$. Then for $b \in B_2'$, $A_s'^b \subseteq A_s'T_{11} \subseteq M^*$. As was shown above, this implies $b \in M^*$. Thus $B_2' \subseteq M^*$. Since $A \subseteq M^*$, M^* contains the inverse image of $\tilde{A}_2'\tilde{B}_2'\tilde{A}_2'$. This is impossible as M^* is solvable. Therefore $\tilde{G} = \tilde{B}_1' \times \tilde{A}_2'\tilde{B}_2'\tilde{A}_2'$. It follows from 6.9 that G is factorizable, a contradiction. This completes the proof of the lemma.

LEMMA 8.4. *G possesses a normal subgroup K such that $|G/K| = 2^s$, $s \geqq 1$.*

Proof. Let $M^* = M(a^*)$. By 8.3, M^* is solvable. Hence $M^* = (AT)B_0$ where $T = (B \cap T)^A \lhd M^*$. Since $N^*(a^*) \nsubseteq N$, $N_{M^*}(A) \subset M^*$ and $T \neq 1$. By 8.3, b_0 is the only involution in B_0. Hence, by 6.10, $N_G(T) \subset G$. Since M^* is a maximal subgroup of G, $N_G(T) = M^*$. By 5.6, $|T| = |B \cap T|^2$. By 5.4(v), T is an elementary abelian 2-group. By 8.3, B_0 is a 2-group. Therefore $S = TB_0$ is a Sylow 2-subgroup of M^*. By 5.4(i). $C_T(b_0) = B \cap T$. Therefore $C_{T\langle b_0 \rangle}(t) = T$ for all $t \in T - (B \cap T)$.

Case 1. $|T/B \cap T| \neq 2$. Suppose $T' \subseteq S$ is elementary abelian and $|T'| = |T|$. Then $TT'/T \subseteq \Omega_1(S/T)$. Since b_0 is the only involution in B_0 and $B_0 \cap T \subseteq C_B(A) = 1$, $\Omega_1(S/T) = T\langle b_0 \rangle / T$. Therefore $T' \subseteq T\langle b_0 \rangle$ and $|T'|/|T \cap T'| = |TT'/T| \leqq 2$. In particular, $|T'| \leqq 2|T \cap T'|$. Since $|T| = |B \cap T|^2 \neq 1$ and $|T/B \cap T| \neq 2$, $2|B \cap T| < |T| = |T'|$. Therefore $|B \cap T| < |T \cap T'|$. Thus there exists an element $t \in T \cap T$ such that $t \notin B \cap T$. Then $T' \subseteq C_{T\langle b_0 \rangle}(t') = T$. Since $|T'| = |T|$, $T' = T$. Thus T is the only elementary abelian subgroup of S of order $|T|$. Hence T char S. Therefore

$N_G(S) \subseteq N_G(T) \subseteq M^*$. Hence S is a Sylow 2-subgroup of G. By Grün's Theorem [4, Theorem 7.4.2],

$$S \cap [G, G] = \langle S \cap [N_G(S), N_G(S)], S \cap [S^x, S^x] \mid x \in G \rangle.$$

Since $N_G(S) \subseteq M^*$ and M^*/AT is abelian,

$$[S, S] \subseteq S \cap [N_G(S), N_G(S)] \subseteq S \cap AT = T.$$

Therefore

$$S \cap [G, G] \subseteq \langle T, S \cap T^x \mid x \in G \rangle. \tag{1}$$

Suppose there exists an element $x \in G$ such that $S \cap T^x \nsubseteq T$. Then there exist elements $u \in S - T$ and $t \in T$ such that $u = x^{-1}tx$. Let $x = aba'$, $a, a' \in A$, $b \in B$. Let $t' = a^{-1}ta$ and $u' = a'ua'^{-1}$. Then $u' = a'(x^{-1}tx)a'^{-1} = b^{-1}t'b$, $t' \in T^a = T$, and $u' \in (S-T)^{a'^{-1}} \subseteq M^* - T$. Now $B \cap T \subseteq C_G(b^{-1}t'b) = C_G(u')$, and $C_{M^*}(B \cap T) = C_A(T)TB_0$. Therefore $u' = a_1t_1b_1$ with $a_1 \in C_A(T)$, $t_1 \in T$, and $b_1 \in B_0$. Hence

$$b_1 a_1^{b_1} t_1^{b_1} t_1 a_1 b_1 = a_1 t_1 b_1 t_1 a_1 b_1 = a_1 t_1 b_1 a_1 t_1 b_1 = u'^2 = b^{-1} t'^2 b = 1.$$

Therefore $b_1^{-2} = a_1^{b_1} t_1^{b_1} t_1 a_1 \in B_0 \cap AT \subseteq B_0 \cap T = 1$. Since $A \cap T = 1$, $a^{b_1}a_1 = t^{b_1}t_1 = 1$. Suppose $b_1 = 1$. Then $a_1^2 = 1$ and therefore $a_1 = 1$. Then $u' = a_1t_1b_1 = t_1 \in T$, a contradiction. Therefore $b_1 \neq 1$. Since b_0 is the only involution in B_0, $b_1 = b_0$. Therefore, since $t_1^{b_1} = t_1$, $t_1 \in C_T(b_0) = B \cap T$. Hence $t' = bu'b^{-1} = ba_1t_1b_1b^{-1} = ba_1b^{-1}t_1b_1$. In particular, $ba_1b^{-1} \in S$. Since $|A|$ is odd, $a_1 = 1$. Therefore $t' = bu'b^{-1} = t_1b_1 = u'$. Hence $u' \in T$, a contradiction. Therefore $S \cap T^x \subseteq T$ for all $x \in G$. Together with (1) this implies $S \cap [G, G] \subseteq T$.

G possesses a normal subgroup K such that $G/K \cong S/S \cap [G, G]$ [4, Theorem 7.3.5]. Since S is a 2-group and $S \cap [G, G] \subseteq T \neq S$, $|G/K| = 2^s$, $s \geqq 1$.

Case 2. $|T/B \cap T| = 2$. Then $|T| = |B \cap T|^2 = 4$. Suppose $b \in B_0$ is of order 4. Then $b^2 \in C_G(T)$. But b^2 is an involution, b_0 is the only involution in B_0, and $b_0 \notin C_G(T)$. Therefore B_0 does not contain any elements of order 4. Hence $B_0 = \langle b_0 \rangle$ and therefore $S = T\langle b_0 \rangle$. It follows that $Z(S) = B \cap T$.

Let P be a Sylow 2-subgroup of G containing S. Then $Z(P) \subseteq N_G(T) \cap P = M^* \cap P = S$. Therefore $Z(P) \subseteq Z(S) = B \cap T$. Since $|B \cap T| = 2$ and $Z(P) \neq 1$, $Z(P) = B \cap T$. Let $t \in T - (B \cap T)$. Then $C_P(t)$ normalizes T. Therefore $C_P(t) = C_S(t) = T$. Since P is nonabelian, $|P/[P, P]| \geqq |P/\phi(P)| \geqq 4$. Now t has $|P|/|C_P(t)| = |P|/|T| = |P|/4$ distinct conjugates $t_1, t_2, \ldots, t_m$ in P. Since $1, t_1t^{-1}, t_2t^{-1}, \ldots, t_mt^{-1}$, are distinct elements of $[P, P]$, $|[P, P]| \geqq |P|/4$. Therefore $|P/[P, P]| = 4$. Hence P is either dihedral, semidihedral, or generalized quaternion [4, Theorem 5.4.5]. Since P contains more than one involution, it is not generalized quaternion.

Since $C_G(B \cap T) \supseteq S$, it is not abelian. By 7.2, $C_G(b_0) = B$ and therefore $C_G(b_0)$ is abelian. Hence G has at least two conjugate classes of involutions. Furthermore, since G is not solvable, G does not possess a normal 2-complement. Therefore G has a normal subgroup K of index 2 [4, Theorem 7.7.3 and Exercise 7.7].

We can now easily complete the proof of Theorem 8.1. By 8.4, G has a normal subgroup K of index $2^s \geqq 2$. Since $|A|$ is odd, $A \subseteq K$. Therefore $K = A(B \cap K)A$. Let $H = K\langle b_0 \rangle$. Then $H \lhd G$ and $H = A(B \cap H)A$. Since $b_0 \in H$, H is of type III. If $H \subset G$, then G is factorizable by 6.8. Since this is not the case, $H = G$. In particular, since $K \subset G$, $b_0 \notin K$. By 8.3, b_0 is the only involution in B_0 and $O(B_0) = 1$. Hence $N_{B \cap K}(A) = 1$ and therefore $N_K(A) = A$. By 3.4, K is solvable. Hence G is solvable and therefore G is factorizable, a contradiction. This contradiction completes the proof.

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